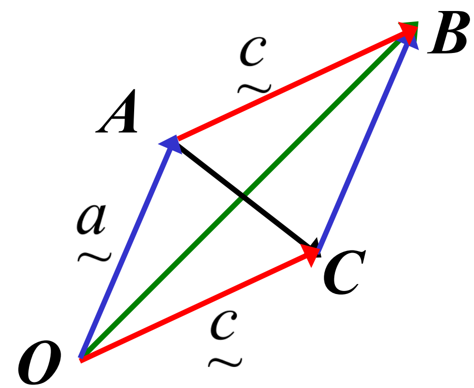
Geometric Proofs

Things to keep in mind when using vectors in a geometric proof

* choose a vertex or other key point to represent the origin
* sum (or difference) of vectors creates a triangle
* parallel vectors are multiples of each other
* the angle between vectors can be found using the dot product
* perpendicular vectors have a dot product equal to zero
* vectors can be written as their position vector using *"head minus tail"*

e.g. Prove that the diagonals of a rhombus are perpendicular



$OABC$ is a rhombus

Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$

Diagonals are OB and AC

$$\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$$

$$\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$$

$$\overrightarrow{OB} \cdot \overrightarrow{AC} = (\underset{\sim}{a} + \underset{\sim}{c}) \cdot (\underset{\sim}{c} - \underset{\sim}{a})$$

$$= \underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{a}$$

$$= |\underset{\sim}{c}|^2 - |\underset{\sim}{a}|^2$$

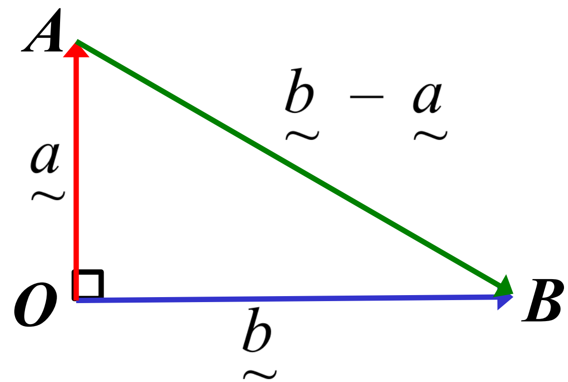
however $|\underset{\sim}{a}| = |\underset{\sim}{c}|$ (sides in a rhombus are =)

so $\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$

$\therefore OB \perp AC$

(ii) Prove Pythagoras' Theorem

OAB is a right angled triangle



Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$

$$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$$

$$OA^2 + OB^2$$

$$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$$

$$AB^2 = |\underset{\sim}{b} - \underset{\sim}{a}|^2$$

$$= (\underset{\sim}{b} - \underset{\sim}{a})\cdot(\underset{\sim}{b} - \underset{\sim}{a})$$

$$= \underset{\sim}{b}\cdot\underset{\sim}{b} - 2\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{a}$$

however $OA \perp OB$

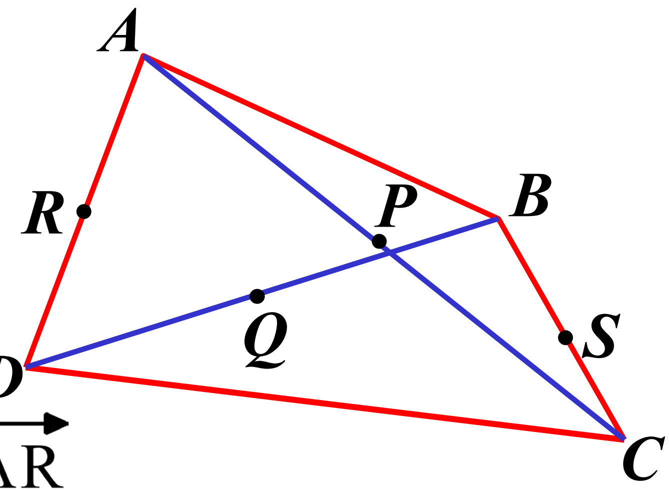
so $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$$AB^2 = \underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{a}$$

$$= |\underset{\sim}{b}|^2 + |\underset{\sim}{a}|^2$$

$$\therefore AB^2 = OA^2 + OB^2$$

(*iii*) *ABCD* is a quadrilateral, *P, Q, R* and *S* are midpoints of the lines *AC, BD, AD* and *BC* respectively.
What type of quadrilateral is *PRQS*?



treat *A* as the origin

Let $\overrightarrow{AB} = \underset{\sim}{u}$, $\overrightarrow{AC} = \underset{\sim}{v}$, $\overrightarrow{AD} = \underset{\sim}{w}$

$$\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\underset{\sim}{v}$$

$$\overrightarrow{BQ} = \frac{1}{2}\overrightarrow{BD} = \frac{1}{2}(\underset{\sim}{w} - \underset{\sim}{u})$$

$$\overrightarrow{AR} = \frac{1}{2}\overrightarrow{AD} = \frac{1}{2}\underset{\sim}{w}$$

$$\overrightarrow{BS} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u})$$

$$\overrightarrow{PR} = \overrightarrow{PA} + \overrightarrow{AR}$$

$$= -\frac{1}{2}\underset{\sim}{v} + \frac{1}{2}\underset{\sim}{w} = \frac{1}{2}(\underset{\sim}{w} - \underset{\sim}{v})$$

$$\overrightarrow{SQ} = \overrightarrow{SB} + \overrightarrow{BQ}$$

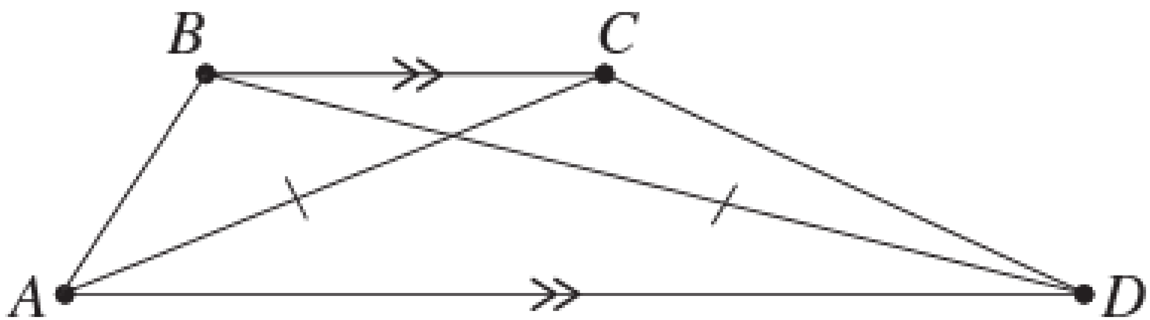
$$= \frac{1}{2}(\underset{\sim}{u} - \underset{\sim}{v}) + \frac{1}{2}(\underset{\sim}{w} - \underset{\sim}{u}) = \frac{1}{2}(\underset{\sim}{w} - \underset{\sim}{v})$$

$$\therefore \overrightarrow{PR} = \overrightarrow{SQ}$$

Thus *PRQS* is a parallelogram as a pair of opposite sides are both equal and parallel

(iv) **2021 Extension 1 HSC Q14c) (ii)**

In the trapezium $ABCD$, BC is parallel to AD and $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{BD}\right|$



Let $\underset{\sim}{a} = \overrightarrow{AB}$, $\underset{\sim}{b} = \overrightarrow{BC}$ and $\overrightarrow{AD} = k\overrightarrow{BC}$, where $k > 0$

Show that $2\underset{\sim}{a}\cdot\underset{\sim}{b} + (1 - k)|\underset{\sim}{b}|^2 = 0$

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} \qquad\qquad \overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$$

$$= \underset{\sim}{a} + \underset{\sim}{b} \qquad\qquad = -\underset{\sim}{a} + k\underset{\sim}{b}$$

$$\left|\overrightarrow{AC}\right|^2 = \left|\overrightarrow{BD}\right|^2$$

$$(\underset{\sim}{a} + \underset{\sim}{b})\cdot(\underset{\sim}{a} + \underset{\sim}{b}) = (k\underset{\sim}{b} - \underset{\sim}{a})\cdot(k\underset{\sim}{b} - \underset{\sim}{a})$$

$$|\underset{\sim}{a}|^2 + 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2 = k^2|\underset{\sim}{b}|^2 - 2k\underset{\sim}{b}\cdot\underset{\sim}{a} + |\underset{\sim}{a}|^2$$

$$|\underset{\sim}{a}|^2 + 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2 = k^2|\underset{\sim}{b}|^2 - 2k\underset{\sim}{b}\cdot\underset{\sim}{a} + |\underset{\sim}{a}|^2$$

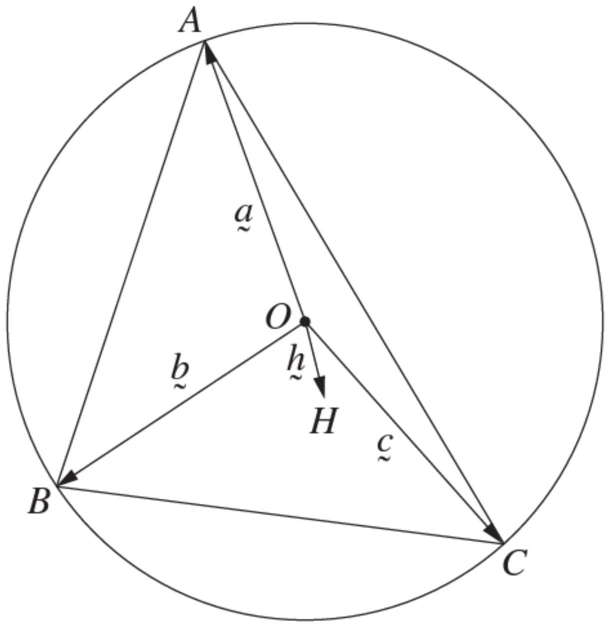
$$2(k+1)\underset{\sim}{a}\cdot\underset{\sim}{b} + (1-k^2)|\underset{\sim}{b}|^2 = 0$$

$$2(k+1)\underset{\sim}{a}\cdot\underset{\sim}{b} + (1-k)(1+k)|\underset{\sim}{b}|^2 = 0$$

$$2\underset{\sim}{a}\cdot\underset{\sim}{b} + (1-k)|\underset{\sim}{b}|^2 = 0 \qquad (k > 0 \therefore 1+k \neq 0)$$

(v) 2022 Extension 1 HSC Q13a)

Three different points A, B and C are chosen on a circle centred at O. Let $\underset{\sim}{a} = \overrightarrow{OA}$, $\underset{\sim}{b} = \overrightarrow{OB}$ and $\underset{\sim}{c} = \overrightarrow{OC}$. Let $\underset{\sim}{h} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ and let H be the point such that $\overrightarrow{OH} = \underset{\sim}{h}$, as shown in the diagram.



Show that $\overrightarrow{BH}$ and $\overrightarrow{CA}$ are perpendicular.

$\overrightarrow{BH} = \underset{\sim}{h} - \underset{\sim}{b}$ $\overrightarrow{CA} = \underset{\sim}{a} - \underset{\sim}{c}$

$$\overrightarrow{BH} \cdot \overrightarrow{CA} = (\underset{\sim}{h} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{c})$$

$$= (\underset{\sim}{a} + \underset{\sim}{c}) \cdot (\underset{\sim}{a} - \underset{\sim}{c})$$

$$= a \cdot a \ - c \cdot c$$

$$= |\underset{\sim}{a}|^2 - |\underset{\sim}{c}|^2$$

$$= 0 \qquad \left(|\underset{\sim}{a}| = |\underset{\sim}{c}| \ , \ = \quad \text{radii} \right)$$

$$\therefore \overrightarrow{BH} \perp \overrightarrow{CA}$$

**Exercise 8D; 1 to 4, 6 to 8,
10, 11, 12**