

# ***$n$ th Roots Of Unity ( $z^n = \pm 1$ )***

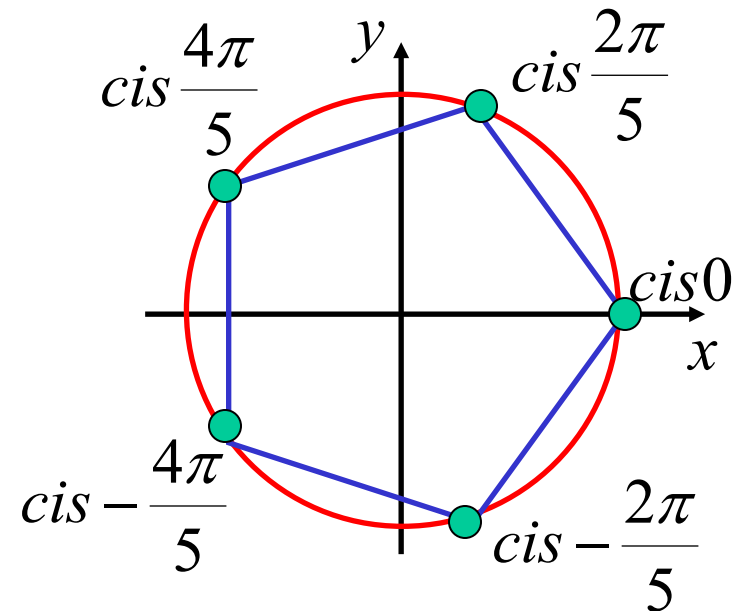
The solutions of equations of the form,  $z^n = \pm 1$ , are the  $n$ th roots of unity

When placed on an Argand Diagram, they form a regular  $n$  sided polygon, with vertices on the unit circle.

e.g.  $z^5 = 1$

$$z = \text{cis} \left[ \frac{2\pi k + 0}{5} \right] \quad k = 0, \pm 1, \pm 2$$

$$z = \text{cis} 0, \text{cis} \frac{2\pi}{5}, \text{cis} -\frac{2\pi}{5}, \text{cis} \frac{4\pi}{5}, \text{cis} -\frac{4\pi}{5}$$



b) (i) If  $\omega$  is a complex root of  $z^5 - 1 = 0$ , show that  $\omega^2, \omega^3, \omega^4$  and  $\omega^5$  are the other roots.

$$z^5 = 1 \quad \text{If } \omega \text{ is a solution then } \omega^5 = 1$$

$$\therefore (\omega^5)^5 = 1^5$$

$$= 1 \quad \therefore \omega^5 \text{ is a solution}$$

A complex root  
means a non-real  
root

$$(\omega^2)^5 = (\omega^5)^2$$

$$= 1^2$$

$$= 1$$

$$(\omega^3)^5 = (\omega^5)^3$$

$$= 1^3$$

$$= 1$$

$$(\omega^4)^5 = (\omega^5)^4$$

$$= 1^4$$

$$= 1$$

$\therefore \omega^2$  is a solution       $\therefore \omega^3$  is a solution       $\therefore \omega^4$  is a solution

Thus if  $\omega$  is a root then  $\omega^2, \omega^3, \omega^4$  and  $\omega^5$  are also roots

**OR**

$$z^5 = 1 \quad \text{If } \omega \text{ is a solution then } \omega^5 = 1$$

$$\text{If } k \in \mathbb{Z} \text{ then } (\omega^k)^5 = (\omega^5)^k$$

$$= 1^k = 1$$

Thus if  $\omega$  is a root then  $\omega^2, \omega^3, \omega^4$  and  $\omega^5$  are also roots

(ii) Prove that  $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

$$\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 = -\frac{b}{a} \quad (\text{sum of the roots})$$

$$\underline{1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0} \quad (\omega^5 = 1)$$

**OR**  $\omega^5 - 1 = 0$

$$(\omega - 1)(1 + \omega + \omega^2 + \omega^3 + \omega^4) = 0$$

$$\underline{\therefore 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0} \quad (\omega \neq 1)$$

$$\left[ \begin{array}{l} \text{NOTE :} \\ \omega^n - 1 = 0 \\ (\omega - 1)(1 + \omega + \omega^2 + \dots + \omega^{n-1}) = 0 \end{array} \right]$$

**OR**

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 = \frac{a(r^n - 1)}{r - 1} \quad \text{GP: } a = 1, r = \omega, n = 5$$
$$= \frac{1(\omega^5 - 1)}{\omega - 1} = \underline{0} \quad (\omega^5 - 1 = 0)$$

(iii) Find the quadratic equation whose roots are  $\omega + \omega^4$  and  $\omega^2 + \omega^3$

$$\begin{aligned} \alpha + \beta &= \omega + \omega^4 + \omega^2 + \omega^3 \\ &= -1 \end{aligned}$$

$$\begin{aligned} \alpha\beta &= (\omega + \omega^4)(\omega^2 + \omega^3) \\ &= \omega^3 + \omega^4 + \omega^6 + \omega^7 \\ &= \omega^3 + \omega^4 + \omega + \omega^2 \\ &= -1 \end{aligned}$$

$$\underline{\therefore \text{equation is } x^2 + x - 1 = 0}$$

c) If  $\omega$  is a complex cube root of unity, use the fact that  $1 + \omega + \omega^2 = 0$  to;

(i) Evaluate  $(1 + \omega^2)^3$

$$= (-\omega)^3$$

$$= -\omega^3$$

$$= \underline{-1}$$

(ii) Evaluate  $\frac{1}{1 + \omega} + \frac{1}{1 + \omega^2}$

$$= -\frac{1}{\omega^2} - \frac{1}{\omega}$$

$$= \frac{-1 - \omega}{\omega^2}$$

$$= \frac{\omega^2}{\omega^2}$$

$$= \underline{1}$$

(iii) Form the cubic equation with roots  $1, 1 + \omega, 1 + \omega^2$

$$(z - 1)\{z^2 - (2 + \omega + \omega^2)z + (1 + \omega + \omega^2 + \omega^3)\} = 0$$

$$(z - 1)(z^2 - z + 1) = 0$$

$$z^3 - z^2 + z - z^2 + z - 1 = 0$$

$$\underline{z^3 - 2z^2 + 2z - 1 = 0}$$

d) Solve  $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$

Now  $z^6 - 1 = (z - 1)(z^5 + z^4 + z^3 + z^2 + z + 1)$

And  $z^6 - 1 = 0$  has solutions;

$$z = \text{cis} \left[ \frac{2\pi k}{6} \right] \quad k = 0, \pm 1, \pm 2, 3$$

$$z^5 + z^4 + z^3 + z^2 + 1 = 0$$

$$\frac{(z - 1)(z^5 + z^4 + z^3 + z^2 + 1)}{(z - 1)} = 0$$

$$z^6 - 1 = 0, z \neq 1$$

$$z = \text{cis} \frac{\pi}{3}, \text{cis} -\frac{\pi}{3}, \text{cis} \frac{2\pi}{3}, \text{cis} -\frac{2\pi}{3}, \text{cis} \pi$$

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -1$$

---

e) 1996 HSC

$$\text{Let } \omega = \cos \frac{2\pi}{9} + i \sin \frac{2\pi}{9}$$

(i) Show that  $\omega^k$  is a solution of  $z^9 - 1 = 0$ , where  $k$  is an integer

$$z^9 = 1$$

$$z = \text{cis} \left[ \frac{2\pi k}{9} \right] \quad k = 0, 1, 2, 3, 4, 5, 6, 7, 8$$

$$z = \left\{ \text{cis} \frac{2\pi}{9} \right\}^k$$

$$\underline{z = \omega^k}$$

(ii) Prove that  $\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = -1$

$$z^9 - 1 = 0$$

$$\therefore 1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = 0 \quad (\text{sum of roots})$$

$$\underline{\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = -1}$$

(iii) Hence show that  $\cos \frac{\pi}{9} \cos \frac{2\pi}{9} \cos \frac{4\pi}{9} = \frac{1}{8}$

$$\begin{aligned}
 & z^9 - 1 \\
 &= (z-1)(z-\omega)(z-\omega^8)(z-\omega^2)(z-\omega^7)(z-\omega^3)(z-\omega^6)(z-\omega^4)(z-\omega^5) \\
 &= (z-1) \left( z^2 - 2 \cos \frac{2\pi}{9} z + 1 \right) \left( z^2 - 2 \cos \frac{4\pi}{9} z + 1 \right) \\
 &\quad \left( z^2 - 2 \cos \frac{6\pi}{9} z + 1 \right) \left( z^2 - 2 \cos \frac{8\pi}{9} z + 1 \right) \\
 &= (z-1) \left( z^2 - 2 \cos \frac{2\pi}{9} z + 1 \right) \left( z^2 - 2 \cos \frac{4\pi}{9} z + 1 \right) \\
 &\quad (z^2 + z + 1) \left( z^2 - 2 \cos \frac{8\pi}{9} z + 1 \right)
 \end{aligned}$$

Let  $z = i$

$$i^9 - 1 = (i-1) \left( -2 \cos \frac{2\pi}{9} i \right) \left( -2 \cos \frac{4\pi}{9} i \right) (i) \left( -2 \cos \frac{8\pi}{9} i \right)$$

$$i^9 - 1 = (i - 1) \left( -2 \cos \frac{2\pi}{9} i \right) \left( -2 \cos \frac{4\pi}{9} i \right) (i) \left( -2 \cos \frac{8\pi}{9} i \right)$$

$$i - 1 = -i^4 (i - 1) \left( 2 \cos \frac{2\pi}{9} \right) \left( 2 \cos \frac{4\pi}{9} \right) \left( 2 \cos \frac{8\pi}{9} \right)$$

$$-1 = 8 \cos \frac{2\pi}{9} \cos \frac{4\pi}{9} \cos \frac{8\pi}{9}$$

$$-1 = 8 \cos \frac{2\pi}{9} \cos \frac{4\pi}{9} \left( -\cos \frac{\pi}{9} \right)$$

$$\cos \frac{\pi}{9} \cos \frac{2\pi}{9} \cos \frac{4\pi}{9} = \frac{1}{8}$$

---



**f) 2022 Extension 2 HSC Q13c)**

Consider the equation  $z^5 + 1 = 0$ , where  $z$  is a complex number

(i) Solve the equation  $z^5 + 1 = 0$  by finding the 5<sup>th</sup> roots of  $-1$

$$z^5 + 1 = 0$$

$$z^5 = -1$$

$$z = \text{cis}\left(\frac{\pi + 2\pi k}{5}\right) \quad k = 0, \pm 1, \pm 2$$

$$\underline{z = -1, \text{cis}\left(\pm\frac{\pi}{5}\right), \text{cis}\left(\pm\frac{3\pi}{5}\right)}$$

(ii) Show that if  $z$  is a solution of  $z^5 + 1 = 0$  and  $z \neq -1$ , then  $u = z + \frac{1}{z}$  is a solution of  $u^2 - u - 1 = 0$

$$\begin{aligned}u^2 - u - 1 &= \left(z + \frac{1}{z}\right)^2 - \left(z + \frac{1}{z}\right) - 1 \\&= z^2 + 2 + \frac{1}{z^2} - z - \frac{1}{z} - 1 \\&= z^2 + 2 - z^3 - z + z^4 - 1 && (z^5 = -1) \\&= z^4 - z^3 + z^2 - z + 1 \\&= \frac{(z + 1)(z^4 - z^3 + z^2 - z + 1)}{(z + 1)} && (z \neq -1) \\&= \frac{z^5 + 1}{z + 1} \\&= 0 \\&\quad \therefore \underline{u = z + \frac{1}{z} \text{ is a solution of } u^2 - u - 1 = 0}\end{aligned}$$

(iii) Hence find the exact value of  $\cos\frac{3\pi}{5}$

$$\text{let } z = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$$

$$u^2 - u - 1 = 0$$

$$z + \frac{1}{z} = 2\cos\frac{3\pi}{5}$$

$$u = \frac{1 \pm \sqrt{5}}{2}$$

from (ii) as  $z$  is a solution of  $z^5 + 1 = 0$  then  $u = z + \frac{1}{z}$  is a solution to  $u^2 - u - 1 = 0$

$$\text{but } \cos\frac{3\pi}{5} < 0$$

$$\therefore 2\cos\frac{3\pi}{5} = \frac{1 - \sqrt{5}}{2}$$

$$\cos\frac{3\pi}{5} = \frac{1 - \sqrt{5}}{4}$$

**Exercise 3C; 1 to 4, 5ac, 6, 7, 8, 9, 11, 12, 13**