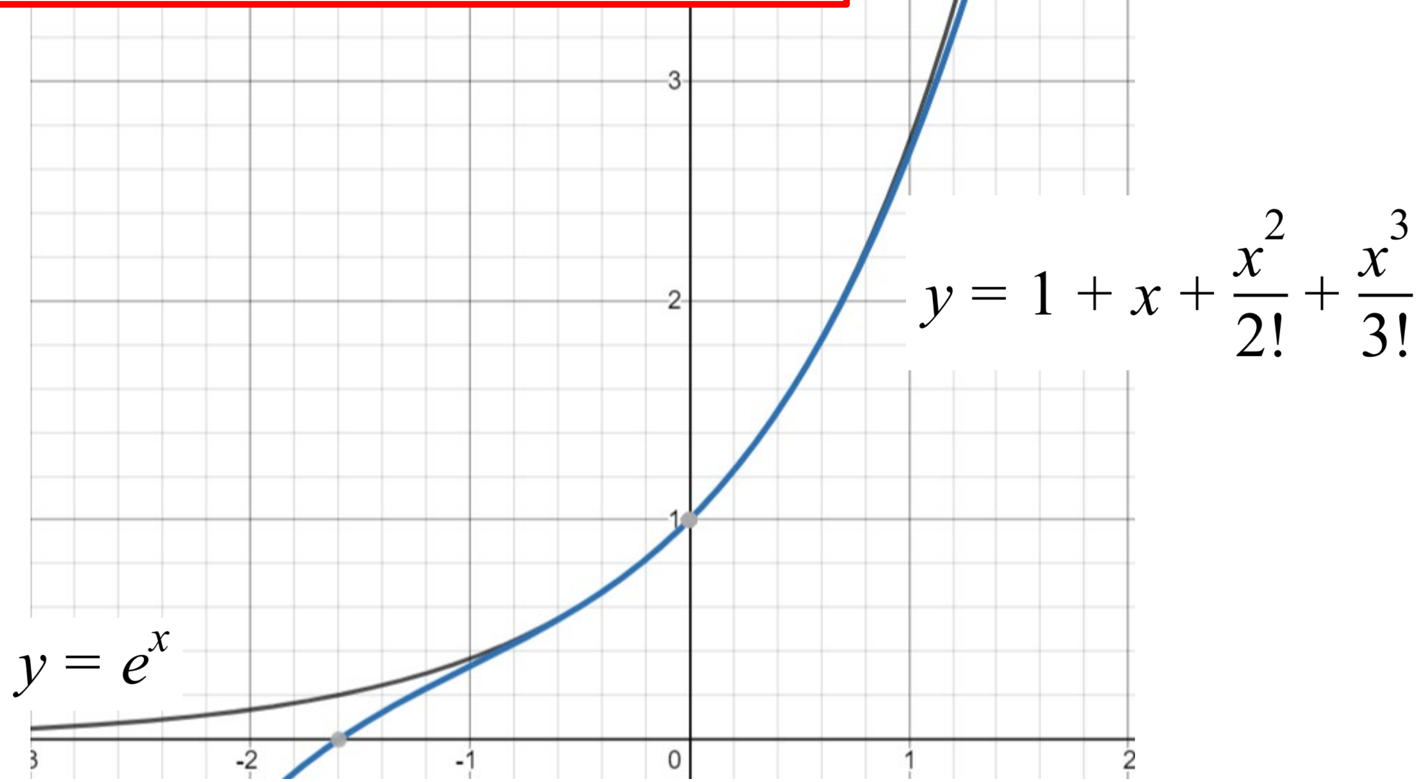
Euler's Formula

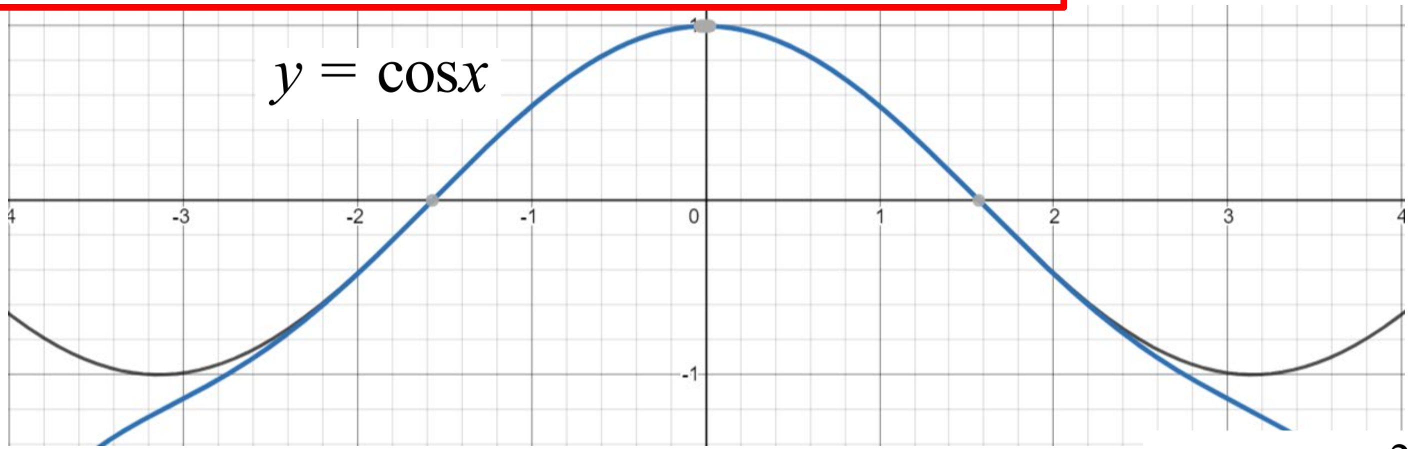
Maclaurin Series

A Maclaurin series is a representation of a function as an infinite sum

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

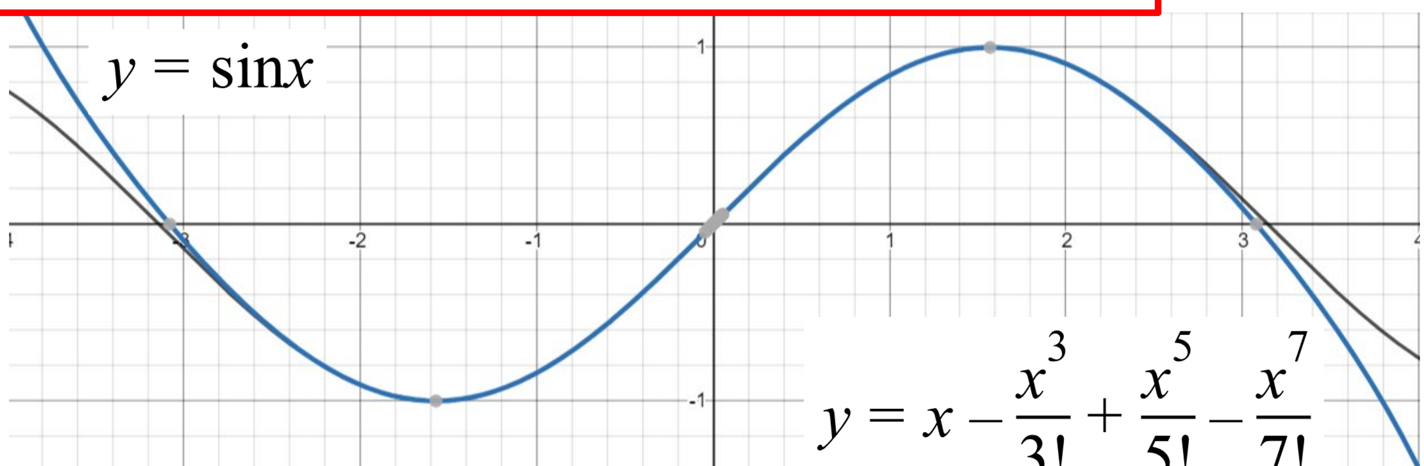


$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$



$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$



Euler's Formula

Euler's formula uses imaginary numbers to convert between exponential and trigonometric functions

Let $x = \theta$ in the Maclaurin series expressions for $\cos x$ and $\sin x$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots \qquad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$$

Let $x = i\theta$ in the expression for e^x

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$$

$$= \cos\theta + i\sin\theta$$

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

for real θ

This now gives a third way for expressing complex numbers

$$z = x + iy \; = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Cartesian form **polar form** **exponential form**

e.g. Write $4 - 4i$ in exponential form

$$4 - 4i = 4\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$$

$$= 4\sqrt{2}\,e^{-\frac{i\pi}{4}}$$

Using Euler's Formula for multiplication and division

e.g. if $z = \sqrt{3} + i$ and $w = 1 - \sqrt{3}$ i, find;

$$(i)\ zw = (\sqrt{3} + i)(1 - \sqrt{3}\, i)$$

$$= 2e^{\frac{i\pi}{6}} \times 2e^{-\frac{i\pi}{3}}$$

$$= 4e^{-\frac{i\pi}{6}}$$

$$= 4\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$= 2\sqrt{3} - 2i$$

$$(ii)\ \frac{z}{w} = \frac{2e^{\frac{i\pi}{6}}}{2e^{-\frac{i\pi}{3}}}$$

$$= e^{\frac{i\pi}{2}}$$

$$= i$$

$$(iii)\ z^5 = 2^5\, e^{\frac{5i\pi}{6}}$$

$$= 32\left(\frac{-\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$= -16\sqrt{3} + 16i$$

(*iv*) **2022 Extension 2 HSC Q12e)**

Given the complex number $z = e^{i\theta}$, show that $w = \dfrac{z^2 - 1}{z^2 + 1}$ is purely imaginary

$$w = \frac{z^2 - 1}{z^2 + 1}$$

$$= \frac{e^{2i\theta} - 1}{e^{2i\theta} + 1}$$

$$= \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$$

$e^{i\theta}$ and $e^{-i\theta}$ are conjugates

$$= \frac{2i\sin\theta}{2\cos\theta}$$

$= i\tan\theta$ which is purely imaginary

Exercise 3D;1ac, 2cd, 3abf, 4ade, 5, 6bd, 7, 8, 11ad, 12ab (i, ii, iv), 14bd