Applications of Euler's Formula

Exponential Functions

$$
\begin{aligned}
e^{i\theta + 2ik\pi} &= \cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k) \\
&= \cos\theta + i\sin\theta \text{ if } k \in \mathbb{Z} \\
&= e^{i\theta}
\end{aligned}
$$

$$e^{i(\theta + 2\pi k)} = e^{i\theta}$$

So e^z is a periodic function, with period $2\pi i$

Trigonometric Functions

Earlier we proved the following identities:

$$z^n + \frac{1}{z^n} = 2\cos n\theta \text{ and } z^n - \frac{1}{z^n} = 2i\sin n\theta$$

rearranging these we get;

$$2\cos n\theta = e^{n\theta i} + e^{-n\theta i} \qquad 2i\sin n\theta = e^{n\theta i} - e^{-n\theta i}$$

$$\cos n\theta = \frac{1}{2}\left(e^{n\theta i} + e^{-n\theta i}\right) \qquad \sin n\theta = \frac{1}{2i}\left(e^{n\theta i} - e^{-n\theta i}\right)$$

e.g. Prove the identities;

(i) $\sin^2\theta + \cos^2\theta = 1$

$$\sin^2\theta + \cos^2\theta = \frac{1}{(2i)^2}\left(e^{i\theta} - e^{-i\theta}\right)^2 + \frac{1}{2^2}\left(e^{i\theta} + e^{-i\theta}\right)^2$$

$$= \frac{1}{4}\left(-e^{2i\theta} + 2 - e^{-2i\theta} + e^{2i\theta} + 2 + e^{-2i\theta}\right)$$

$$= \frac{1}{4}(4) = \underline{1}$$

$$(ii)\ \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

$$\sin^3\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^3$$

$$= \frac{e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}}{-8i}$$

$$= \frac{1}{4}\left(\frac{3\left(e^{i\theta} - e^{-i\theta}\right) - \left(e^{3i\theta} - e^{-3i\theta}\right)}{2i}\right)$$

$$= \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

$(iii)\ 2\sin\alpha\cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$

$$2\sin\alpha\cos\beta = 2 \times \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \times \frac{e^{i\beta} + e^{-i\beta}}{2}$$

$$= \frac{e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} - e^{-i(\alpha-\beta)} - e^{-i(\alpha+\beta)}}{2i}$$

$$= \frac{e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}}{2i} + \frac{e^{i(\alpha-\beta)} - e^{i(\alpha-\beta)}}{2i}$$

$$= \sin(\alpha + \beta) + \sin(\alpha - \beta)$$

Roots of Complex Numbers

e.g. Find the square roots of $1-\sqrt{3}\,i$

$$z^2 = 1-\sqrt{3}\,i$$
$$= 2e^{-\frac{i\pi}{3}}$$

$$e^z = e^{z+2i\pi k}$$

$$z = \sqrt{2}\,e^{-\frac{i\pi}{6}+\pi k}, k = 0,1$$

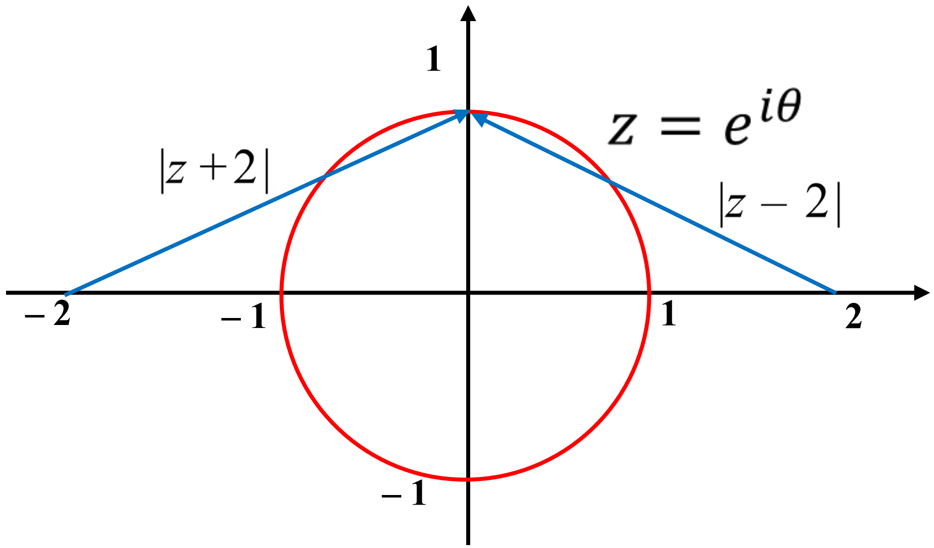
$$z = \sqrt{2}\,e^{-\frac{i\pi}{6}}, \sqrt{2}\,e^{\frac{5i\pi}{6}}$$

$$z = \sqrt{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right), \sqrt{2}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$
$$= \frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}i, \; -\frac{\sqrt{6}}{2}+\frac{1}{2}i$$

e.g. **2020 Extension 2 HSC Q9**

What is the maximum value of $|e^{i\theta} - 2| + |e^{i\theta} + 2|$ for $0 \le \theta \le 2\pi$?

Let $z = e^{i\theta} \Rightarrow |z| = 1$



$|z-2|$ is the length of the vector joining z to 2

$|z+2|$ is the length of the vector joining z to -2

maximum $|z-2| + |z+2|$ is when $z = \pm i$

$$|i - 2| = \sqrt{1^2 + 2^2}$$
$$= \sqrt{5}$$

$\therefore$ max $|z-2| + |z+2| = 2\sqrt{5}$

e.g. **2021 Extension 2 HSC Q14c) (ii)**

(ii) By using part (i), or otherwise, show that $Re\left(e^{\frac{i\pi}{10}}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}$

NOTE: in part (i) you were asked to show $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$$Re\left(e^{\frac{i\pi}{10}}\right) = \cos\frac{\pi}{10}$$

let $\theta = \frac{\pi}{10}$;

$$16\cos^5\left(\frac{\pi}{10}\right) - 20\cos^3\left(\frac{\pi}{10}\right) + 5\cos\left(\frac{\pi}{10}\right) = \cos\frac{\pi}{2} \qquad \text{(from (i))}$$

$$= 0$$

let $x = \cos\frac{\pi}{10}$;

$$16x^5 - 20x^3 + 5x = 0$$

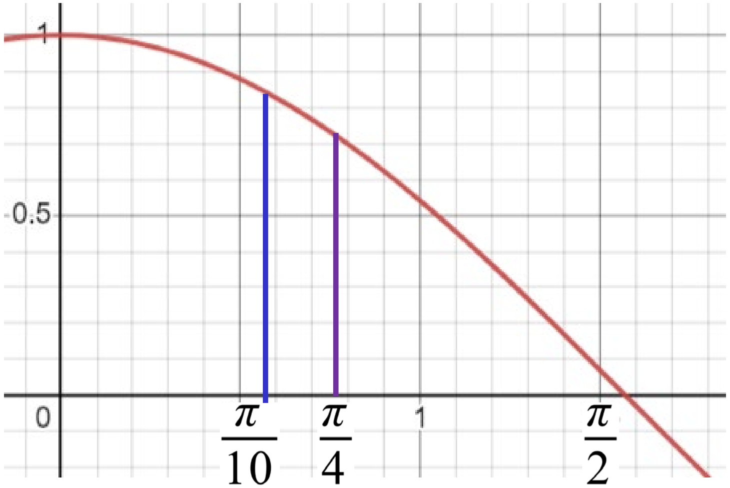
$$16x^4 - 20x^2 + 5 = 0 \qquad (x \neq 0)$$

$$x^2 = \frac{20 \pm \sqrt{320}}{32}$$

$$= \frac{5 \pm \sqrt{5}}{8}$$

$$\therefore \cos\frac{\pi}{10} = \sqrt{\frac{5 \pm \sqrt{5}}{8}} \qquad \left(\text{as } \frac{\pi}{10} \text{ is acute , } \cos\frac{\pi}{10} > 0\right)$$

$$= 0.59 \text{ or } 0.95$$



from the graph we observe

$$\cos\frac{\pi}{10} > \cos\frac{\pi}{4} = 0.71$$

$$\therefore \cos\frac{\pi}{10} = \sqrt{\frac{5 + \sqrt{5}}{8}}$$

$$\text{thus } Re\left(e^{\frac{i\pi}{10}}\right) = \sqrt{\frac{5 + \sqrt{5}}{8}}$$

Exercise 3E;
1, 3, 4bc, 6, 7, 8b ii, iii,11bc, 14