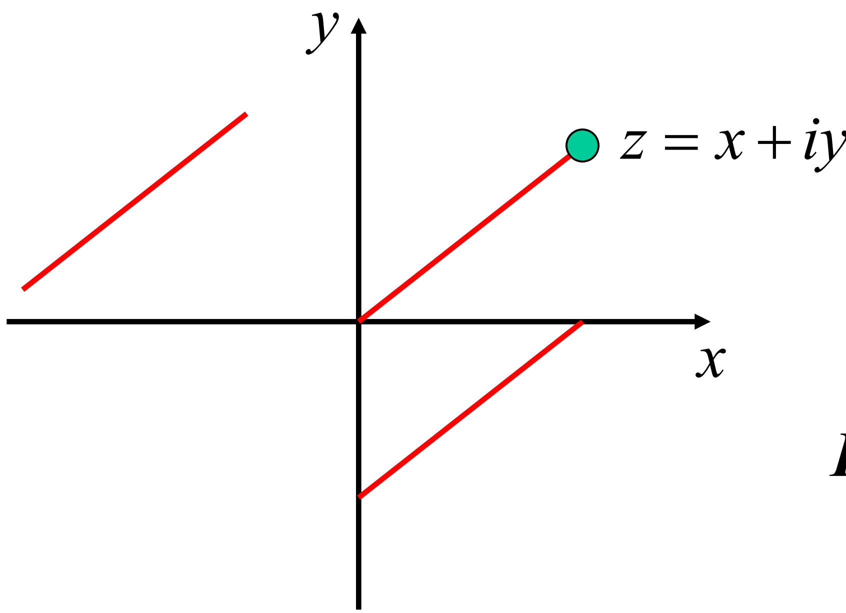
Geometrical Representation of Complex Numbers

Complex numbers can be represented on the Argand Diagram as vectors.

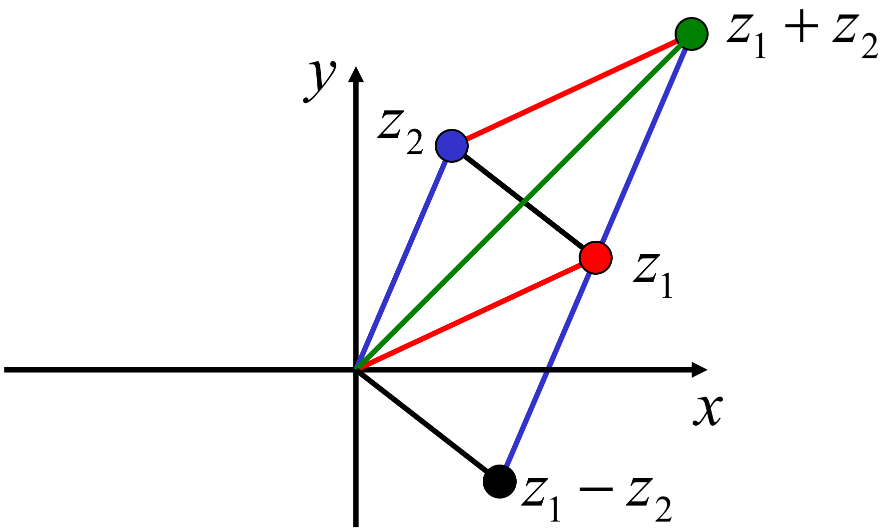


A vector always represents HEAD minus TAIL

The advantage of using vectors is that they can be moved around the Argand Diagram

No matter where the vector is placed its length (modulus) and the angle made with the x axis (argument) is constant

Addition / Subtraction



NOTE :
the parallelogram formed by adding vectors has two diagonals;
$z_1 + z_2$ and $z_1 - z_2$

To add two complex numbers, place the vectors ***"head to tail"***

To subtract two complex numbers, place the vectors ***"head to head"*** (or add the negative vector)

Trianglar Inequality

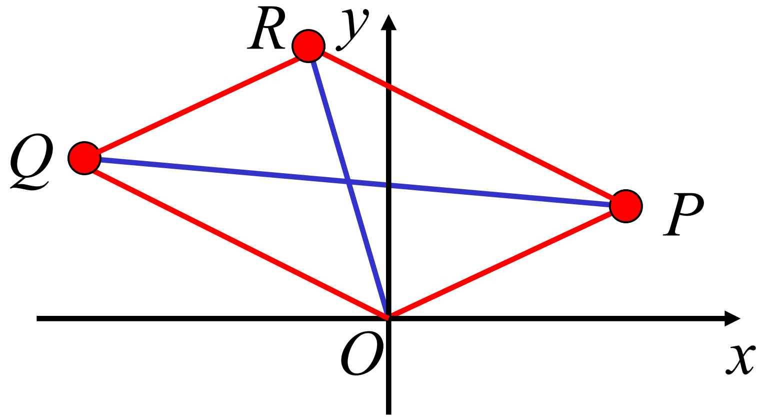
In any triangle a side will be shorter than the sum of the other two sides and bigger than the difference of the other two sides

In ΔABC; $AC \leq AB + BC$ and $AC - AB \leq BC$

(equality occurs when ABC is a straight line)

$$\left||z_1| - |z_2|\right| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

e.g.(1995)



The diagram shows a complex plane with origin O.
The points P and Q represent the complex numbers z and w respectively.
Thus the length of PQ is $|z-w|$

(i) Show that $|z-w| \leq |z| + |w|$

The length of OP is $|z|$

The length of OQ is $|w|$

The length of PQ is $|z-w|$

Using the triangular inequality on ΔOPQ

$$|z-w| \leq |z| + |w|$$

(ii) Construct the point R representing $z + w$, What can be said about the quadrilateral $OPRQ$?

$OPRQ$ is a parallelogram

(iii) If $|z-w|=|z+w|$, what can be said about $\frac{w}{z}$?

$|z-w|=|z+w|$ i.e. diagonals in $OPRQ$ are =

$\therefore OPRQ$ is a rectangle

$$\arg w - \arg z = \frac{\pi}{2}$$

$$\arg \frac{w}{z} = \frac{\pi}{2}$$

$\therefore \frac{w}{z}$ is purely imaginary

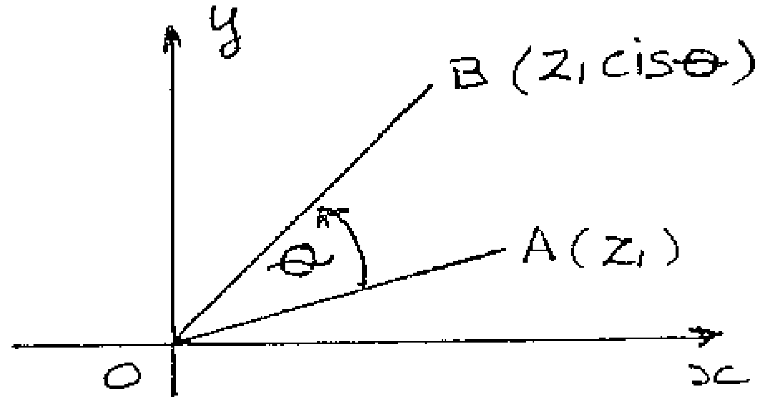
Multiplication

$|z_1 z_2| = |z_1||z_2|$ $\arg z_1 z_2 = \arg z_1 + \arg z_2$

$$r_1 cis\theta_1 \times r_2 cis\theta_2 = r_1 r_2 cis(\theta_1 + \theta_2)$$

i.e. if we multiply z_1 by z_2, the vector z_1 is rotated anticlockwise by θ_2 and its length is multiplied by r_2

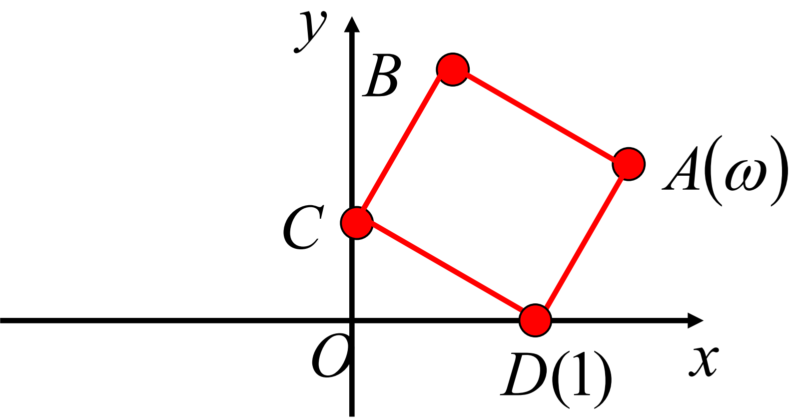
If we multiply z_1 by $cis\theta$ the vector OA will rotate by an angle of θ in an anti-clockwise direction. If we multiply by $rcis\theta$ it will also multiply the length of OA by a factor of r



Note: $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=i \quad \therefore iz_1$ will rotate OA anticlockwise 90 degrees.

Multiplication by i is a rotation anticlockwise by $\frac{\pi}{2}$

REMEMBER: ***A vector is HEAD minus TAIL***



$$\overrightarrow{DC} = \overrightarrow{DA} \times i$$
$$C - 1 = (\omega - 1)i$$
$$C = 1 + (\omega - 1)i$$
$$= (1 - i) + i\omega$$

OR

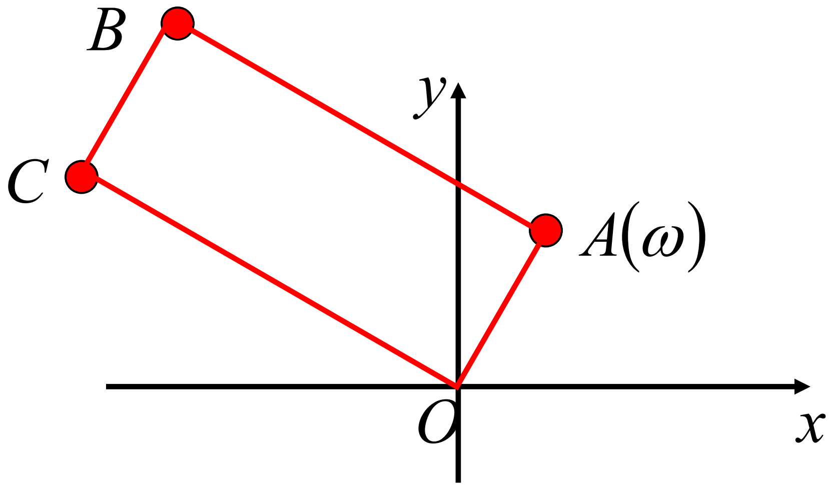
$$B = A + \overrightarrow{DC}$$
$$B = \omega + C - 1$$
$$B = \omega + (\omega - 1)i$$
$$= -i + (1 + i)\omega$$

OR

$$B = C + \overrightarrow{DA}$$
$$B = (1 - i) + i\omega + (\omega - 1)$$
$$= -i + (1 + i)\omega$$

$$\overrightarrow{DB} = \sqrt{2}cis\frac{\pi}{4} \times \overrightarrow{DA}$$
$$B - 1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)(\omega - 1)$$
$$B = (1 + i)(\omega - 1) + 1$$
$$= \omega - 1 + i\omega - i + 1$$
$$= -i + (1 + i)\omega$$

e.g.(2000)



In the Argand Diagram, $OABC$ is a rectangle, where $OC = 2OA$.
The vertex A corresponds to the complex number ω

(i) What complex number corresponds to C?

$$\overrightarrow{OC} = \overrightarrow{OA} \times 2i$$

$$C = 2i\omega$$

(ii) What complex number corresponds to the point of intersection D of the diagonals OB and AC?

diagonals bisect in a rectangle

$\therefore D =$ midpoint of AC

$$D = \frac{A+C}{2}$$

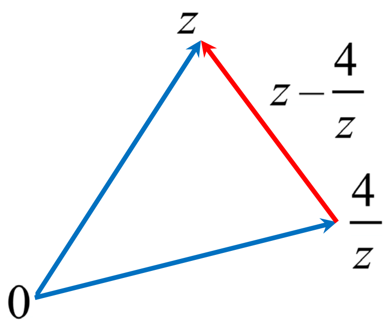
$$D = \frac{\omega + 2i\omega}{2}$$

$$\therefore D = \left(\frac{1}{2} + i\right)\omega$$

(*ii*) 2022 Extension 2 HSC Q15d)

The complex number z satisfies $\left|z-\frac{4}{z}\right|=2$

Using the triangle inequality, or otherwise, show that $|z|\leq\sqrt{5}+1$



by the Δ inequality

$$|z|\leq\left|z-\frac{4}{z}\right|+\left|\frac{4}{z}\right|$$

$$=2+\frac{4}{|z|}$$

$$|z|^2\leq 2|z|+4$$

$$|z|^2-2|z|-4\leq 0$$

$$(|z|-1)^2\leq 5$$

$$-\sqrt{5}\leq|z|-1\leq\sqrt{5}$$

$$1-\sqrt{5}\leq|z|\leq 1+\sqrt{5}$$

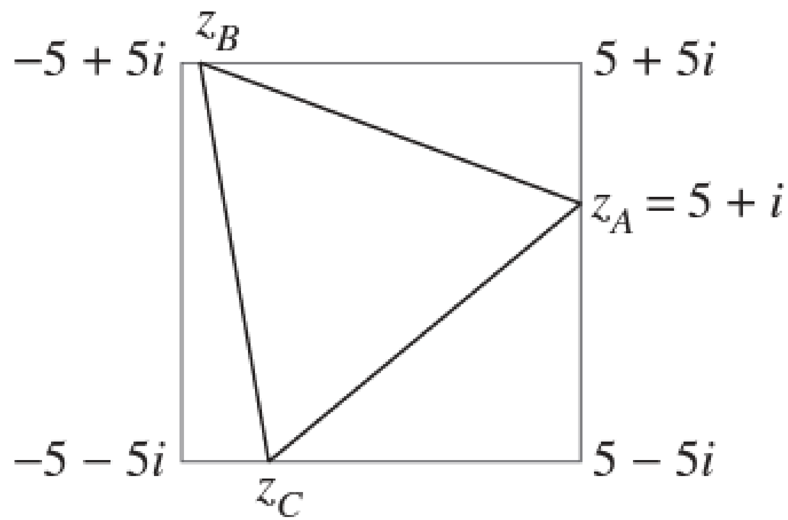
$$\therefore |z|\leq 1+\sqrt{5}$$

(*iii*) 2022 Extension 2 HSC Q16a)

A square in the Argand plane has vertices

$$5 + 5i,\ 5 - 5i,\ -5 - 5i \text{ and } -5 + 5i$$

The complex numbers $z_A = 5 + i$, z_B and z_C lie on the square and form the vertices of an equilateral triangle, as shown in the diagram.



Find the exact value of the complex number z_B

$$z_B = b + 5i \qquad \underset{\sim}{u} = z_B - z_A \qquad \underset{\sim}{v} = z_C - z_A$$

$$z_C = c - 5i \qquad = (b-5) + 4i \qquad = (c-5) - 6i$$

$$\underset{\sim}{v} = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\underset{\sim}{u}$$

$$(c-5) - 6i = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right)[(b-5) + 4i]$$

$$= \frac{1}{2}(b-5) + 2i + \frac{\sqrt{3}}{2}(b-5)i - 2\sqrt{3}$$

equating imaginary parts

$$-6 = 2 + \frac{\sqrt{3}}{2}(b-5)$$

$$\frac{\sqrt{3}}{2}(b-5) = -8$$

$$\sqrt{3}\,b - 5\sqrt{3} = -16$$

$$b = 5 - \frac{16}{\sqrt{3}}$$

$$\therefore z_B = 5 - \frac{16}{\sqrt{3}} + 5i$$

Exercise 1E; 2 to 9, 11, 12, 13, 16 to 19, 21, 23, 24