Locus and Complex Numbers

$\omega = f(z)$, find the locus of ω or z
given some condition for ω or z
(Make the condition the subject)

$$\omega \text{ is purely real} \Rightarrow \operatorname{Im}(\omega) = 0, \arg \omega = 0 \text{ or } \pi$$

$$\omega \text{ is purely imaginary} \Rightarrow \operatorname{Re}(\omega) = 0, \arg \omega = \pm \frac{\pi}{2}$$

$$\arg\left(\frac{\text{linear function}}{\text{linear function}}\right) = \theta \Rightarrow \text{locus is an arc of a circle}$$

$$*\text{ minor arc if } \theta > \frac{\pi}{2}$$

$$*\text{ major arc if } \theta < \frac{\pi}{2}$$

$$*\text{ semicircle if } \theta = \frac{\pi}{2}$$

e.g. (i) Find the locus of w if $w=\dfrac{z+2}{z}$, $|z|=4$

$$w=\frac{z+2}{z}$$

$$zw=z+2$$

$$z(w-1)=2$$

$$z=\frac{2}{(w-1)}$$

$$\therefore\left|\frac{2}{(w-1)}\right|=4$$

$$\frac{2}{|w-1|}=4$$

$$|w-1|=\frac{1}{2}$$

$\therefore$ locus is a circle, centre $(1,0)$ and radius $\dfrac{1}{2}$

i.e. $(x-1)^2+y^2=\dfrac{1}{4}$

(ii) Find the locus of z if $w = \dfrac{z+1}{z-1}$ and w is purely real

$$w = \frac{(x+1)+iy}{(x-1)+iy} \times \frac{(x-1)-iy}{(x-1)-iy}$$

$$= \frac{(x^2-1) - i(x+1)y + i(x-1)y + y^2}{(x-1)^2 + y^2}$$

If w is purely real then $\text{Im}(w) = 0$

$$i.e. -(x+1)y + (x-1)y = 0$$

$$-xy - y + xy - y = 0$$

$$-2y = 0$$

$$y = 0$$

$\therefore$ locus is $y = 0$, excluding $(1,0)$

$(z - 1 \neq 0$, bottom of fraction $\neq 0)$

Note: locus is $y = 0$, excluding $(1,0)$ only

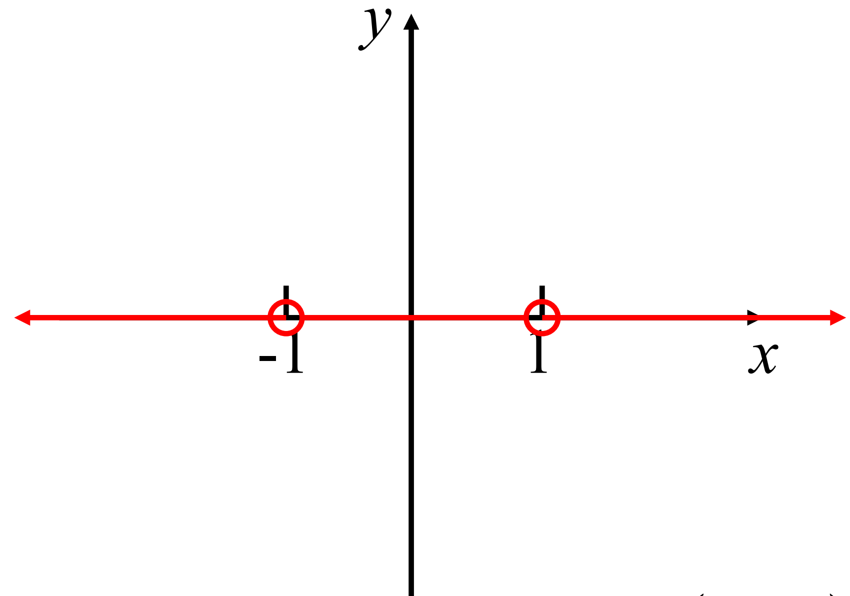
i.e. answer the original question

OR

If w is purely real then $\arg w = 0$ or π

$$i.e. \arg\left(\frac{z+1}{z-1}\right) = 0 \text{ or } \pi$$

$$\arg(z+1) - \arg(z-1) = 0 \text{ or } \pi$$



locus is $y = 0$, excluding $(\pm 1, 0)$

(iii) Find the locus of z if $\arg\left(\frac{z}{z-4}\right)=\frac{\pi}{6}$

$$\arg\left(\frac{z}{z-4}\right)=\frac{\pi}{6}$$

$$\arg z-\arg(z-4)=\frac{\pi}{6}$$

$$\frac{y}{2}=\tan 60$$

$$y=2\tan 60$$

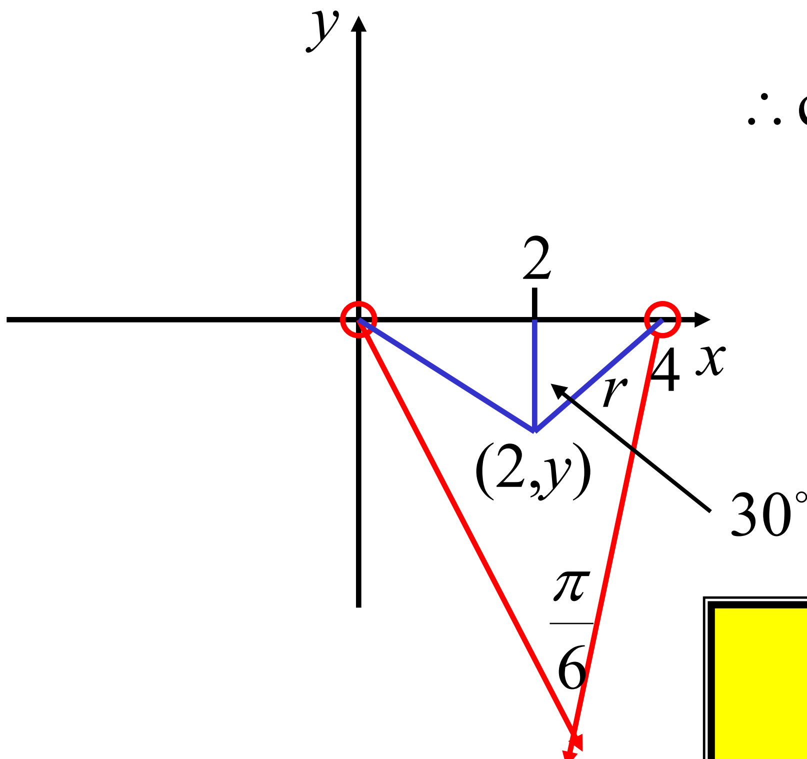
$$=2\sqrt{3}$$

$$r^2=2^2+\left(2\sqrt{3}\right)^2$$

$$r^2=16$$

$$r=4$$

$\therefore$ centre is $\left(2,-2\sqrt{3}\right)$



$\therefore$ locus is the major arc of the circle $(x-2)^2+\left(y+2\sqrt{3}\right)^2=16$ formed by the chord joining $(0,0)$ and $(4,0)$ but not including these points.

NOTE: $\arg z>\arg(z-4)$

$\therefore$ below axis

Exercise 1F; 8, 9, 14, 16, 17, 19, 20

Patel: **Exercise 4N; 5, 6**

HSC Geometrical Complex Numbers Questions