Polynomials & Complex Numbers

If a polynomial's coefficients are all real then the roots will appear in complex conjugate pairs.

Every polynomial of degree *n* can be;

- factorised as a mixture of quadratic and linear factors over the real field
- factorised to *n* linear factors over the complex field

NOTE: odd ordered polynomials must have a real root

$$e.g.\ (i)\ x^2+2x+2 = (x+1)^2+1$$

$$= (x+1+i)(x+1-i)$$

$(ii)\ z^4 + z^2 - 12 = 0$

$$(z^2 - 3)(z^2 + 4) = 0$$

$$(z + \sqrt{3})(z - \sqrt{3})(z^2 + 4) = 0 \qquad \text{(factorised over Real numbers)}$$

$$(z + \sqrt{3})(z - \sqrt{3})(z + 2i)(z - 2i) = 0 \qquad \text{(factorised over Complex numbers)}$$

$$\underline{z = \pm\sqrt{3} \quad \text{or} \quad z = \pm 2i}$$

> If $(x - a)$ is a factor of $P(x)$, then $P(a) = 0$
>
> If $(ax - b)$ is a factor of $P(x)$, then $P\left(\frac{b}{a}\right) = 0$

(iii) Factorise $2x^3 - 3x^2 + 8x + 5$

as it is a cubic it must have a real factor

$$P\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 + 8\left(-\frac{1}{2}\right) + 5$$

$$= -\frac{1}{4} - \frac{3}{4} - 4 + 5$$

$$= 0$$

$$\therefore 2x^3 - 3x^2 + 8x + 5$$

$$= (2x + 1)(x^2 - 2x + 5)$$

$$= (2x + 1)\left[(x - 1)^2 + 4\right]$$

$$\underline{= (2x + 1)(x - 1 - 2i)(x - 1 + 2i)}$$

(iv) Given that $P(x) = 4x^4 + 8x^3 + 5x^2 + x - 3$ has two rational zeros, find these zeros and factorise $P(x)$ over the complex field.

$$P\left(\frac{1}{2}\right) = 4\left(\frac{1}{16}\right) + 8\left(\frac{1}{8}\right) + 5\left(\frac{1}{4}\right) + \frac{1}{2} - 3$$

$$= 0$$

$$\therefore (2x-1) \text{ is a factor}$$

$$P(x) = (2x-1)(2x^3 + 5x^2 + 5x + 3)$$

$$P\left(-\frac{3}{2}\right) = 2\left(\frac{-27}{8}\right) + 5\left(\frac{9}{4}\right) + 5\left(-\frac{3}{2}\right) + 3$$

$$= 0$$

$$\therefore (2x+3) \text{ is a factor}$$

$$\therefore \text{rational zeros are } \frac{1}{2} \text{ and } -\frac{3}{2}$$

$$P(x) = 4x^4 + 8x^3 + 5x^2 + x - 3$$
$$= (2x-1)(2x+3)(x^2 + x \qquad + 1)$$
$$= (2x-1)(2x+3)\left[\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}\right]$$
$$= (2x-1)(2x+3)\left(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(x+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$$

$$1 \times 3 \times 2x = 6x$$
$$1 \times 2x \times -1 = -2x$$
$$4x$$
$$-1 \times 3 \times ?\, x = -3x$$

(v) Solve $z^4 - 6z^3 + 14z^2 - 30z + 45 = 0$ given that it has a real root of multiplicity 2

real roots must be a divisor of 45 i.e. 1, 3, 5, 9, 15, 45

$$P'(z) = 4z^3 - 18z^2 + 28z - 30$$

$P'(1) \neq 0$ $\qquad$ $P'(3) = 4(3)^3 - 18(3)^2 + 28(3) - 30 = 0$

$$P(3) = (3)^4 - 6(4)^3 + 14(3)^2 - 30(3) + 45 = 0$$

$$\therefore (z-3)^2 \text{ is a factor}$$

$$z^4 - 6z^3 + 14z^2 - 30z + 45 = 0$$

$$(z-3)^2(z^2 \qquad + 5) = 0$$

$$(z-3)^2(z - \sqrt{5}\, i)(z + \sqrt{5}\, i) = 0$$

$$\underline{z = 3 \text{ or } z = \pm\sqrt{5}\, i}$$

(vi) **2022 Extension 2 HSC Q16d)**

Find all the complex numbers z_1, z_2 and z_3 that satisfy the following three conditions simultaneously.

$$|z_1| = |z_2| = |z_3|$$

$$z_1 + z_2 + z_3 = 1$$

$$z_1 \ z_2 \ z_3 = 1$$

$$|z_1 \ z_2 \ z_3| = 1$$

$$|z_3|^3 = 1$$

$$|z_3| = 1$$

$$z_1 \ z_2 = \frac{1}{z_3}$$

$$= \frac{\overline{z_3}}{|z_3|^2}$$

$$= \overline{z_3}$$

similarly $z_1 \ z_3 = \overline{z_2}$ and $z_2 \ z_3 = \overline{z_1}$

$$z_1 \ z_2 + z_1 \ z_3 + z_2 \ z_3 = \overline{z_3} + \overline{z_2} + \overline{z_1}$$

$$= \overline{z_3 + z_2 + z_1}$$

$$= 1$$

z_1, z_2, z_3 are the solutions to

$$x^3 - (z_1 + z_2 + z_3)x^2 + (z_1 z_2 + z_1 z_3 + z_2 z_3)x - z_1 z_2 z_3 = 0$$

$$x^3 - x^2 + x - 1 = 0$$

$$(x - 1)(x^2 + 1) = 0$$

$$x = 1 \quad \text{or} \quad x = \pm i$$

$$\therefore z_1 = 1 \;,\; z_2 = i \;,\; z_3 = -i$$

Exercise 1G; 1b, 2, 3, 5, 6, 7b, 8 to 19, 22a to c

***Patel:* Exercise 5B, 6b, 7b, 8acegh**