

Algebraic Inequalities

Properties of Inequalities

If $a > b$ then $a \pm x > b \pm x$

you can add or subtract the same amount to both sides

If $a > b$ then $ax > bx$ for $x > 0$

you can multiply or divide both sides by a positive amount

If $a > b$ then $ax < bx$ for $x < 0$

the inequality sign changes when you multiply or divide both sides by a negative amount

If $a > b > 0$ then $\frac{1}{a} < \frac{1}{b}$

when both sides are positive, the inequality sign changes when you take the reciprocal of both sides

If $a > b > 0$ then $a^2 > b^2$

when both sides are positive, you can square both sides

If $a > b$ and $b > c$ then $a > c$

inequalities of the same sign can be linked

If $a > b > 0$ and $c > d > 0$
then $ac > bd$

when both sides are positive, inequalities of the same sign can be multiplied

If $a > b > 0$ and $c > d > 0$
then $a + c > b + d$

when both sides are positive, inequalities of the same sign can be added

Inequality Techniques

To prove $x \geq y$, it can be easier to prove $x - y \geq 0$

e.g. (i) (1995) Prove $pq \leq \frac{p^2 + q^2}{2}$

$$\begin{aligned}\frac{p^2 + q^2}{2} - pq &= \frac{p^2 - 2pq + q^2}{2} \\ &= \frac{(p - q)^2}{2} \\ &\geq 0\end{aligned}$$

$$\therefore \underline{\frac{p^2 + q^2}{2} \geq pq}$$

OR Assume $pq > \frac{p^2 + q^2}{2}$

$$2pq > p^2 + q^2$$

$$0 > p^2 - 2pq + q^2$$

$$0 > (p - q)^2$$

But $(p - q)^2 > 0$ which is a contradiction

$$\therefore \underline{pq \leq \frac{p^2 + q^2}{2}}$$

Start with a known result

(ii) (1994) a) Prove $a^2 + b^2 + c^2 > ab + bc + ac$

$$(a - b)^2 > 0$$

$$a^2 - 2ab + b^2 > 0$$

$$\therefore a^2 + b^2 > 2ab$$

$$a^2 + c^2 > 2ac$$

$$b^2 + c^2 > 2bc$$

$$2a^2 + 2b^2 + 2c^2 > 2ab + 2ac + 2bc$$

$$\underline{a^2 + b^2 + c^2 > ab + ac + bc}$$

b) If $a + b + c = 1$, prove $ab + ac + bc < \frac{1}{3}$

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc)$$

$$\therefore (a + b + c)^2 - 2(ab + ac + bc) > ab + ac + bc$$

$$3(ab + ac + bc) < (a + b + c)^2$$

$$3(ab + ac + bc) < 1$$

$$\underline{ab + ac + bc < \frac{1}{3}}$$

Substitute different expressions into known inequalities

c) Prove $\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$

$$a^2 + b^2 + c^2 \geq ab + ac + bc$$

$$a^2 + b^2 + c^2 - ab - ac - bc \geq 0$$

$$(a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc) \geq 0$$

$$a^3 + ab^2 + ac^2 - a^2b - a^2c - abc + a^2b + b^3 + bc^2 - ab^2 - abc - b^2c \\ + a^2c + b^2c + c^3 - abc - ac^2 - bc^2 \geq 0$$

$$a^3 + b^3 + c^3 - 3abc \geq 0$$

$$\frac{1}{3}(a^3 + b^3 + c^3) \geq abc$$

let $a = a^{\frac{1}{3}}, b = b^{\frac{1}{3}}, c = c^{\frac{1}{3}}$

$$\frac{1}{3}(a+b+c) \geq a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$$

$$\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$$

Arithmetic Mean \geq Geometric Mean

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

d) Suppose $(1+x)(1+y)(1+z) = 8$, prove $xyz \leq 1$

$$(1+x)(1+y)(1+z) = 8$$

$$1 + x + y + xy + z + xz + yz + xyz = 8$$

$$\frac{1}{3}(x + y + z) \geq \sqrt[3]{xyz} \quad \text{AM} \geq \text{GM}$$

$$\underline{x + y + z \geq 3\sqrt[3]{xyz}}$$

$$xy + yz + xz \geq 3\sqrt[3]{(xy)(yz)(xz)}$$

$$xy + yz + xz \geq 3\sqrt[3]{x^2 y^2 z^2}$$

$$\underline{xy + yz + xz \geq 3\left(\sqrt[3]{xyz}\right)^2}$$

$$1 + x + y + z + xy + xz + yz + xyz = 8$$

$$1 + 3\sqrt[3]{xyz} + 3\left(\sqrt[3]{xyz}\right)^2 + xyz \leq 8$$

$$1 + 3\sqrt[3]{xyz} + 3\left(\sqrt[3]{xyz}\right)^2 + \left(\sqrt[3]{xyz}\right)^3 \leq 8$$

$$\left(1 + \sqrt[3]{xyz}\right)^3 \leq 8$$

$$1 + \sqrt[3]{xyz} \leq 2$$

$$\sqrt[3]{xyz} \leq 1$$

$$\underline{xyz \leq 1}$$

OR

$$(1+x) \geq 2\sqrt{x}$$

AM \geq GM

$$(1+y) \geq 2\sqrt{y}$$

$$(1+z) \geq 2\sqrt{z}$$

$$(1+x)(1+y)(1+z) \geq 2\sqrt{x} \times 2\sqrt{y} \times 2\sqrt{z}$$

$$= 8\sqrt{xyz}$$

$$\therefore 8 \geq 8\sqrt{xyz}$$

$$1 \geq \sqrt{xyz}$$

$$\sqrt{xyz} \leq 1$$

$$\underline{xyz \leq 1}$$

(iii) Prove $\frac{9}{a+b+c} \leq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{a+c} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

$$(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 2\sqrt{ab} \times 2\sqrt{\frac{1}{ab}} \quad (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 3\sqrt[3]{abc} \times 3\sqrt[3]{\frac{1}{abc}}$$

$$\frac{1}{a} + \frac{1}{b} \stackrel{=4}{\geq} \frac{4}{a+b}$$

$$\frac{1}{b} + \frac{1}{c} \geq \frac{4}{b+c}$$

$$\frac{1}{a} + \frac{1}{c} \geq \frac{4}{a+c}$$

$$\frac{2}{a} + \frac{2}{b} + \frac{2}{c} \geq \frac{4}{a+b} + \frac{4}{b+c} + \frac{4}{a+c}$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{a+c}$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \stackrel{=9}{\geq} \frac{9}{a+b+c}$$

Replace a with $a + b$

b with $b + c$

c with $a + c$

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{a+c} \geq \frac{9}{2(a+b+c)}$$

$$\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{a+c} \geq \frac{9}{a+b+c}$$

$$\frac{9}{a+b+c} \leq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{a+c} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

(iv) a) If $a > 0$ is a real number, show that $a + \frac{1}{a} \geq 2$

$$a + \frac{1}{a} \geq 2\sqrt{a \times \frac{1}{a}} \quad \text{AM} \geq \text{GM} \quad \text{OR} \quad \left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 \geq 0$$

$$a + \frac{1}{a} \geq 2\sqrt{1}$$

$$a - 2 + \frac{1}{a} \geq 0$$

$$\underline{a + \frac{1}{a} \geq 2}$$

$$a + \frac{1}{a} \geq 2$$

If $a > 0$, $b > 0$ and $c > 0$;

b) Show that $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$

$$\frac{b}{a} + \frac{a}{b} \geq 2$$

$$\frac{c}{a} + \frac{a}{c} \geq 2$$

$$\frac{c}{b} + \frac{b}{c} \geq 2$$

$$\frac{b}{a} + \frac{a}{b} + \frac{c}{a} + \frac{a}{c} + \frac{c}{b} + \frac{b}{c} \geq 6$$

$$\underline{\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6}$$

c) Show that $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$$

Replace a with $b + c$

b with $a + c$

c with $a + b$

$$\frac{a+c+a+b}{b+c} + \frac{a+b+b+c}{a+c} + \frac{b+c+a+c}{a+b} \geq 6$$

$$\frac{2a}{b+c} + 1 + \frac{2b}{a+c} + 1 + \frac{2c}{a+b} + 1 \geq 6$$

$$\frac{2a}{b+c} + \frac{2b}{a+c} + \frac{2c}{a+b} \geq 3$$

$$\underline{\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}}$$

(v) 2021 Extension 2 HSC Question 15 a)

For all non-negative real numbers x and y , $\sqrt{xy} \leq \frac{x+y}{2}$ (Do NOT prove this)

a) Using this fact, show that for all non-negative real numbers a , b and c ,

$$\sqrt{abc} \leq \frac{a^2 + b^2 + 2c}{4}$$

$$\begin{aligned} \frac{a^2 + b^2}{4} &\geq 2\sqrt{\frac{a^2 b^2}{16}} && \left(\begin{array}{l} \sqrt{xy} \leq \frac{x+y}{2} \\ x+y \geq 2\sqrt{xy} \end{array} \right) && \frac{a^2 + b^2 + 2c}{4} = \frac{a^2 + b^2}{4} + \frac{c}{2} \\ &= \sqrt{\frac{a^2 b^2}{4}} && && \geq \frac{ab}{2} + \frac{c}{2} \\ &= \frac{ab}{2} && && \geq 2\sqrt{\frac{abc}{4}} \\ & && && = \sqrt{abc} \end{aligned}$$

$$\text{thus } \sqrt{abc} \leq \frac{a^2 + b^2 + 2c}{4}$$

b) Using part a), or otherwise, show that for all non-negative real numbers a , b and c

$$\sqrt{abc} \leq \frac{a^2 + b^2 + c^2 + a + b + c}{6}$$

$$\sqrt{abc} \leq \frac{a^2 + b^2 + 2c}{4}$$

similarly

$$\sqrt{abc} \leq \frac{a^2 + c^2 + 2b}{4}$$

$$\sqrt{abc} \leq \frac{b^2 + c^2 + 2a}{4}$$

$$\therefore 3\sqrt{abc} \leq \frac{2a^2 + 2b^2 + 2c^2 + 2a + 2b + 2c}{4}$$

$$\sqrt{abc} \leq \frac{a^2 + b^2 + c^2 + a + b + c}{6}$$

(vi) 2022 Extension 2 HSC Question 16 c)

It is given that for positive numbers $x_1, x_2, x_3, \dots, x_n$ with arithmetic mean A ,

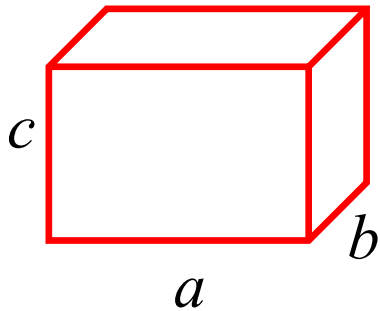
$$\frac{x_1 \times x_2 \times x_3 \times \dots \times x_n}{A^n} \leq 1$$

this is just a confusing way to say $AM > GM$

Suppose that a rectangular prism has dimensions a, b, c and surface area S

(i) Show that $abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$

$$\frac{ab \times ac \times bc}{\left(\frac{ab + ac + bc}{3}\right)^3} \leq 1$$



$$S = 2(ab + ac + bc)$$

$$\begin{aligned} a^2 b^2 c^2 &\leq \left[\frac{2(ab + ac + bc)}{6} \right]^3 \\ &= \left(\frac{S}{6} \right)^3 \end{aligned}$$

$$\therefore abc \leq \left(\frac{S}{6} \right)^{\frac{3}{2}}$$

(ii) Using part (i) , show that when the rectangular prism with surface area S is a cube, it has a maximum volume

If the solid is a cube then $a = b = c$

$$\begin{aligned} abc = a^3 \qquad \left(\frac{S}{6}\right)^{\frac{3}{2}} &= \left[\frac{2(a^2 + a^2 + a^2)}{6}\right]^{\frac{3}{2}} \\ &= (a^2)^{\frac{3}{2}} \\ &= a^3 \end{aligned}$$

so when $a = b = c$

$$\text{volume} = abc = \left(\frac{S}{6}\right)^{\frac{3}{2}}$$

i.e. when the solid is a cube, its volume is at its maximum

(vii) 2023 Extension 2 HSC Question 12 b)

Prove that for all real numbers x and y , where $x^2 + y^2 \neq 0$

$$\frac{(x+y)^2}{x^2+y^2} \leq 2$$

$$\begin{aligned} \frac{(x+y)^2}{x^2+y^2} &= \frac{x^2+2xy+y^2}{x^2+y^2} \\ &= 1 + \frac{2xy}{x^2+y^2} \end{aligned}$$

Now $\frac{x^2+y^2}{2xy} \geq 2 \sqrt{\frac{x^2}{2xy} \times \frac{y^2}{2xy}}$

AM \geq GM

$$= 2 \sqrt{\frac{x^2 y^2}{4x^2 y^2}}$$

$$= 2 \sqrt{\frac{1}{4}}$$

$$= 1$$

$$\frac{x^2 + y^2}{2xy} \geq 1 \Rightarrow \frac{2xy}{x^2 + y^2} \leq 1$$

$$\text{Thus } \frac{(x+y)^2}{x^2 + y^2} \leq 1 + 1$$

$$= 2$$

**Exercise 2D; 2b, 4b,
5a, 8, 11, 12, 13, 14,
15, 16, 19, 20**

$$\begin{aligned} \text{OR } & \frac{(x+y)^2}{x^2 + y^2} - 2 \\ &= \frac{(x+y)^2 - 2(x^2 + y^2)}{x^2 + y^2} \\ &= \frac{x^2 + 2xy + y^2 - 2x^2 - 2y^2}{x^2 + y^2} \\ &= \frac{-x^2 + 2xy - y^2}{x^2 + y^2} \\ &= \frac{-(x-y)^2}{x^2 + y^2} \\ &\leq 0 \\ &\therefore \frac{(x+y)^2}{x^2 + y^2} \leq 2 \end{aligned}$$
