Mathematical Induction

(1) Series Type

e.g. Prove that $\sum_{r=1}^{n} \frac{4}{r(r+1)(r+2)} = 1 - \frac{2}{(n+1)(n+2)}$

Prove the result is true for $n = 1$

$$LHS = \frac{4}{(1)(2)(3)} = \frac{4}{6} = \frac{2}{3}$$

$$RHS = 1 - \frac{2}{(2)(3)} = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\therefore LHS = RHS$$

Hence the result is true for $n = 1$

Assume the result is true for $n = k$, where $k \in \mathbb{Z}^+$

$$\text{i.e. } \sum_{r=1}^{k} \frac{4}{r(r+1)(r+2)} = 1 - \frac{2}{(k+1)(k+2)}$$

Prove the result is true for $n = k + 1$

$$\text{i.e. Prove } \sum_{r=1}^{k+1} \frac{4}{r(r+1)(r+2)} = 1 - \frac{2}{(k+2)(k+3)}$$

Proof:

$$\sum_{r=1}^{k+1} \frac{4}{r(r+1)(r+2)} = \frac{4}{(k+1)(k+2)(k+3)} + \sum_{r=1}^{k} \frac{4}{r(r+1)(r+2)}$$

$$= \frac{4}{(k+1)(k+2)(k+3)} + 1 - \frac{2}{(k+1)(k+2)}$$

$$= 1 - \frac{2(k+3) - 4}{(k+1)(k+2)(k+3)}$$

$$= 1 - \frac{2(k+3)-4}{(k+1)(k+2)(k+3)}$$

$$= 1 - \frac{2k+2}{(k+1)(k+2)(k+3)}$$

$$= 1 - \frac{2(k+1)}{(k+1)(k+2)(k+3)}$$

$$= 1 - \frac{2}{(k+2)(k+3)}$$

It is still a deductive proof, so conclude with the "if then" statement

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

finally, induce the solution

Since the result is true for $n = 1$, then it is true $\forall\, n \in \mathbb{Z}^{+}$ by induction

(ii) 2004 Extension 1 HSC Q4a)

Use mathematical induction to prove that for all integers $n \geq 3$;

$$\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\dots\left(1-\frac{2}{n}\right)=\frac{2}{n(n-1)}$$

Prove the result is true for $n = 3$

$$LHS = 1-\frac{2}{3} \qquad\qquad RHS = \frac{2}{3(2)}$$

$$= \frac{1}{3} \qquad\qquad = \frac{1}{3}$$

$$\therefore LHS = RHS$$

Hence the result is true for $n = 3$

Assume the result is true for $n = k$, where $k \in \mathbb{Z} : k \geq 3$

$$i.e. \left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\dots\left(1-\frac{2}{k}\right)=\frac{2}{k(k-1)}$$

Prove the result is true for $n = k + 1$

$$i.e. \text{ Prove } \left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{4}\right)\left(1 - \frac{2}{5}\right)\cdots\left(1 - \frac{2}{k+1}\right) = \frac{2}{(k+1)k}$$

Proof:

$$\left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{4}\right)\left(1 - \frac{2}{5}\right)\cdots\left(1 - \frac{2}{k+1}\right)$$

$$= \left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{4}\right)\left(1 - \frac{2}{5}\right)\cdots\left(1 - \frac{2}{k}\right)\left(1 - \frac{2}{k+1}\right)$$

$$= \frac{2}{k(k-1)} \times \left(1 - \frac{2}{k+1}\right)$$

$$= \frac{2}{k(k-1)} \times \frac{k+1-2}{(k+1)}$$

$$= \frac{2}{k(k-1)} \times \frac{(k-1)}{(k+1)}$$

$$= \frac{2}{k(k+1)}$$

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 3$, then it is true $\forall\, n \in \mathbb{Z} : n \geq 3$ by induction

(2) Divisibility Type

e.g. Prove that $n^2 + 2n$ is a multiple of 8 if n is even

Prove the result is true for $n = 2$

$$2^2 + 2(2) = 8 \quad \text{, which is divisible by 8}$$

Hence the result is true for $n = 2$

Assume the result is true for $n = k$, where k is an even number

i.e. $k^2 + 2k = 8P$, where $P \in \mathbb{Z} \wedge k$ is even

Prove the result is true for $n = k + 2$

i.e. Prove $(k + 2)^2 + 2(k + 2) = 8Q$, where $Q \in \mathbb{Z}$

Proof:

$$\begin{aligned}(k+2)^2 + 2(k+2) &= k^2 + 4k + 4 + 2k + 4 \\ &= 8P + 4k + 8 \\ &= 8\left(P + \frac{k}{2} + 1\right) \\ &= 8Q \text{ , where } Q = \left(P + \frac{k}{2} + 1\right) \in \mathbb{Z} \text{ as } k \text{ is even}\end{aligned}$$

Hence the result is true for $n = k + 2$ if it is also true for $n = k$

Since the result is true for $n = 2$, then it is true for all even numbers by induction

OR

Assume the result is true for $n = 2k$, where $k \in \mathbb{Z}^+$

i.e. $4k^2 + 4k = 8P$, where $P \in \mathbb{Z}$

Prove the result is true for $n = 2k + 2$

i.e. Prove $(2k + 2)^2 + 2(2k + 2) = 8Q$, where $Q \in \mathbb{Z}$

Proof:

$$
\begin{aligned}
(2k + 2)^2 + 2(2k + 2) &= 4k^2 + 8k + 4 + 4k + 4 \\
&= 8P + 8k + 8 \\
&= 8(P + k + 1) \\
&= 8Q \text{ , where } Q = (P + k + 1) \in \mathbb{Z}
\end{aligned}
$$

Hence the result is true for $n = 2k + 2$ if it is also true for $n = 2k$

Since the result is true for $n = 2$, then it is true for all even numbers by induction

(3) Inequality Type

e.g. (i) Prove $2^n > n^2$, for all integers greater than 4

Prove the result is true for $n = 5$

$LHS = 2^5$ $\quad$ $RHS = 5^2$

$= 32$ $\quad$ $= 25$

$\therefore LHS > RHS$

Hence the result is true for $n = 5$

Assume the result is true for $n = k$, where $k \in \mathbb{Z} : k > 4$

i.e. $2^k > k^2 \wedge k > 4$

Prove the result is true for $n = k + 1$

i.e. Prove $2^{k+1} - (k+1)^2 > 0$

Proof: $2^{k+1} - (k+1)^2 = 2 \times 2^k - k^2 - 2k - 1$

$$> 2k^2 - k^2 - 2k - 1$$

$$= k^2 - 2k - 1$$

$$> 4^2 - 2(4) - 1 \qquad \because\ k > 4$$

$$= 7$$

$$> 0$$

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 5$, then it is true $\forall\ n \in \mathbb{Z} : n > 4$ by induction

(ii) Prove $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$

Prove the result is true for $n = 1$

$$L.H.S = \frac{1}{1^2} = 1 \qquad R.H.S = 2 - \frac{1}{1} = 1$$

$$\therefore L.H.S \leq R.H.S$$

Hence the result is true for $n = 1$

Assume the result is true for $n = k$, where $k \in \mathbb{Z}^+$

$$\text{i.e. } 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$$

Prove the result is true for $n = k + 1$

$$\text{i.e. Prove } 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{(k+1)^2} - 2 + \frac{1}{k+1} \leq 0$$

Proof:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{(k+1)^2} - 2 + \frac{1}{k+1}$$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{k^2} + \frac{1}{(k+1)^2} - 2 + \frac{1}{k+1}$$

$$\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} - 2 + \frac{1}{k+1}$$

$$= \frac{k + k(k+1) - (k+1)^2}{k(k+1)^2}$$

$$= \frac{-1}{k(k+1)^2}$$

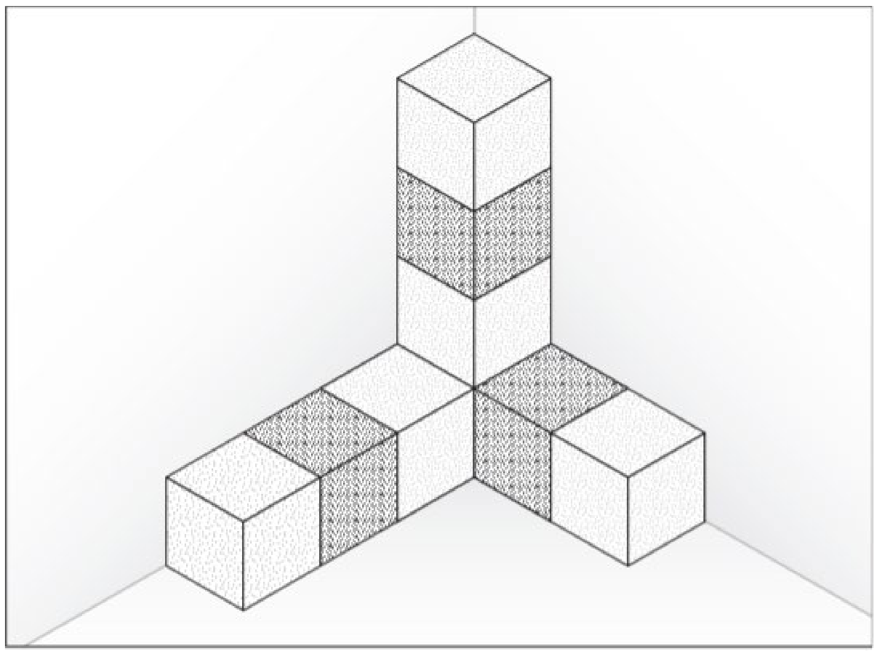
$$< 0 \qquad \because k > 0$$

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 1$, then it is true $\forall\, n \in \mathbb{Z}^+$ by induction

(4) Using the Induction Idea

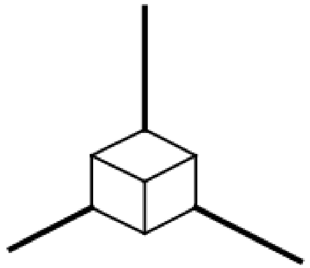
e.g. Two vertical walls and the floor meet at a corner. One cube is placed in the corner. A solid shape is formed by placing identical cubes to form horizontal rows on the floor against the walls or by stacking vertically. An example is the solid shape in the diagram, which is formed from nine cubes.



Let n be the number of cubes used to make a solid shape.

Use mathematical induction to show that the number of exposed faces of the cubes in this shape is $2n + 1$.

Prove the result is true for $n = 1$



when $n = 1$; $2n + 1 = 2(1) + 1$
$= 3$

Hence the result is true for $n = 1$

The first cube will have three exposed faces

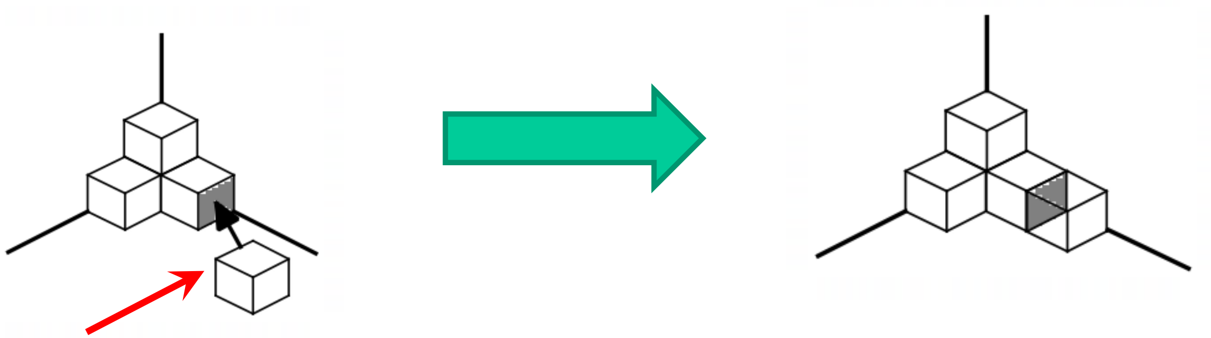
Assume the result is true for $n = k$, where $k \in \mathbb{Z}^+$

i.e. k cubes will have $2k + 1$ exposed faces

Prove true for $n = k + 1$

i.e. $k + 1$ cubes will have $2k + 3$ exposed faces

Proof:



The addition of a new cube will cover up one existing face and expose three faces of the new cube

exposed faces of $k + 1$ cubes = # exposed faces of k cubes – 1 + 3

$$= 2k + 1 - 1 + 3$$

$$= 2k + 3$$

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 1$, then it is true $\forall\, n \in \mathbb{Z}^+$ by induction

Induction & Recursive Formulae

A recursive formula is when one term is defined in terms of one or more preceding terms

If a recursive formula is defined in terms of m preceding terms, you will need to;

1. Prove true for the first m terms
2. Assume true for $n = k$, $n = k - 1$, ... , $n = k - (m - 1)$

Note: *the recursive formula is given to be true, do not try to prove it* e.g. (i) A sequence is defined by; $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$

Prove that $a_n < 2$ for $n \geq 1$

Prove the result is true for $n = 1$

$$a_1 = \sqrt{2} < 2$$

Hence the result is true for $n = 1$

Assume the result is true for $n = k$ where $k \in \mathbb{Z}^+$

i.e. $a_k < 2$

Prove the result is true for $n = k + 1$

i.e. Prove $a_{k+1} < 2$

Proof:

$$\begin{aligned} a_{k+1} &= \sqrt{2 + a_k} \\ &< \sqrt{2+2} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

$$\therefore a_{k+1} < 2$$

Start with the recursive formula

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 1$, then it is true $\forall\, n \in \mathbb{Z}^+$ by induction

(ii) The Fibonacci sequence is defined by;

$$a_1 = a_2 = 1 \qquad a_{n+1} = a_n + a_{n-1} \text{ for } n > 1$$

Prove that $a_n < \left(\frac{1+\sqrt{5}}{2}\right)^n$ for $n \geq 1$

Prove the result is true for $n = 1$ & 2

$\underline{n = 1}$ $\quad L.H.S = a_1 = 1$ $\qquad R.H.S = \left(\frac{1+\sqrt{5}}{2}\right)^1 = 1.62\ldots$

$$\therefore L.H.S \leq R.H.S$$

$\underline{n = 2}$ $\quad L.H.S = a_2 = 1$ $\qquad R.H.S = \left(\frac{1+\sqrt{5}}{2}\right)^2 = 2.62\ldots$

$$\therefore L.H.S \leq R.H.S$$

Hence the result is true for $n = 1$ & 2

Assume the result is true for $n = k$ & $k + 1$ where $k \in \mathbb{Z}^+$

$$\text{i.e. } a_k < \left(\frac{1+\sqrt{5}}{2}\right)^k \wedge a_{k+1} < \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}$$

Prove the result is true for $n = k + 2$

$$\text{i.e. Prove } a_{k+2} - \left(\frac{1+\sqrt{5}}{2}\right)^{k+2} < 0$$

Proof:

$$a_{k+2} - \left(\frac{1+\sqrt{5}}{2}\right)^{k+2} = a_k + a_{k+1} - \left(\frac{1+\sqrt{5}}{2}\right)^{k+2}$$

$$< \left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1+\sqrt{5}}{2}\right)^{k+2}$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^k \left[1 + \frac{1+\sqrt{5}}{2} - \left(\frac{1+\sqrt{5}}{2}\right)^2\right]$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^k > 0$$

$$1+\frac{1+\sqrt{5}}{2}-\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{4+2(1+\sqrt{5})-(1+\sqrt{5})^2}{4}$$

$$= \frac{4+2+2\sqrt{5}-1-2\sqrt{5}-5}{4}$$

$$= 0$$

$$\therefore a_{k+2}-\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} < 0$$

Hence the result is true for $n = k + 2$ if it is also true for $n = k$ & $k + 1$

Since the result is true for $n = 1$ & 2, then it is true $\forall\ n \in \mathbb{Z}^+$ by induction

(iii) **2022 Extension 2 HSC Question 13 b)**

The numbers a_n , for integers $n \geq 1$, are defined as

$$a_1 = \sqrt{2}$$
$$a_2 = \sqrt{2 + \sqrt{2}}$$
$$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}\ \text{, and so on}$$

These numbers satisfy the relation ${a^2}_{n+1} = 2 + a_n$, for $n \geq 1$

Use mathematical induction to prove that $a_n = 2\cos\dfrac{\pi}{2^{n+1}}$, for all integers $n \geq 1$

$$\text{Prove } a_n = 2\cos\frac{\pi}{2^{n+1}},\ \forall n \in \mathbb{Z}^+$$

Prove the result is true for $n = 1$

$$a_1 = 2\cos\frac{\pi}{2^2}$$

$$= 2\cos\frac{\pi}{4}$$

$$= 2 \times \frac{1}{\sqrt{2}}$$

$$= \sqrt{2}$$

Hence the result is true for $n = 1$

Assume the result is true for $n = k$ where $k \in \mathbb{Z}^+$

$$\text{i.e. } a_k = 2\cos\frac{\pi}{2^{k+1}}$$

Prove the result is true for $n = k + 1$

$$\text{i.e. Prove } a_{k+1} = 2\cos\frac{\pi}{2^{k+2}}$$

Proof:

$$a^2_{k+1} = 2 + a_k$$

$$= 2 + 2\cos\frac{\pi}{2^{k+1}}$$

(by assumption)

$$= 4\left[\frac{1}{2}\left(1 + \cos\frac{\pi}{2^{k+1}}\right)\right]$$

$$= 4\left[\frac{1}{2}\left(1 + \cos\left(2 \times \frac{\pi}{2^{k+2}}\right)\right)\right]$$

$$= 4\cos^2\left(\frac{\pi}{2^{k+2}}\right)$$

$$a_{k+1} = 2\cos\frac{\pi}{2^{k+2}}$$

Exercise 2E; 1ce, 2bf, 3a, 4b, 5b, 7, 9d, 10, 12, 13, 15b, 16, 18, 19, 20, 21, 22ac, 23, 24, 25

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 1$, then it is true $\forall\, n \in \mathbb{Z}^+$ by induction