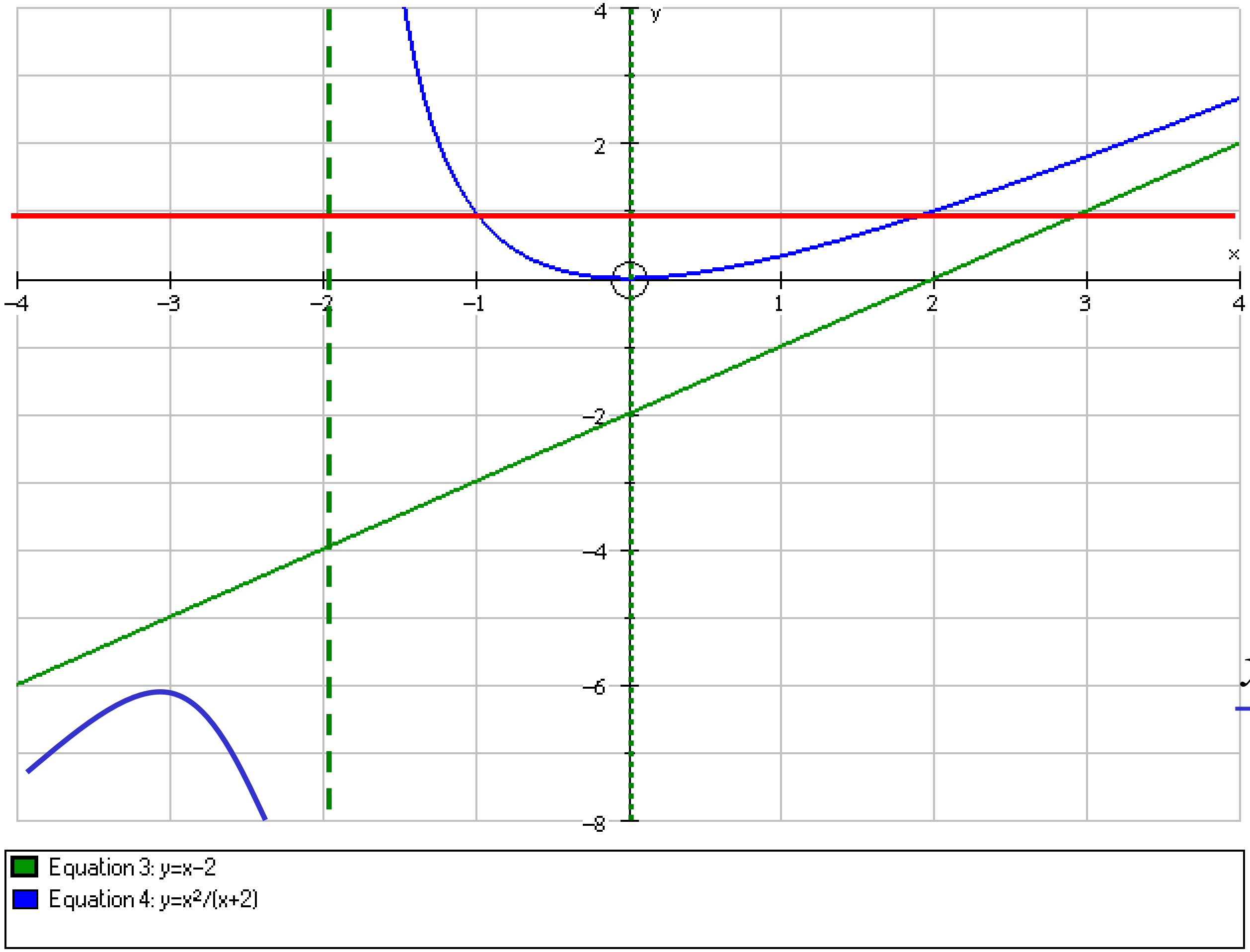
Inequalities & Graphs

e.g. (i) Solve $\dfrac{x^2}{x+2} \leq 1$



$$\frac{x^2}{x+2} = 1$$

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

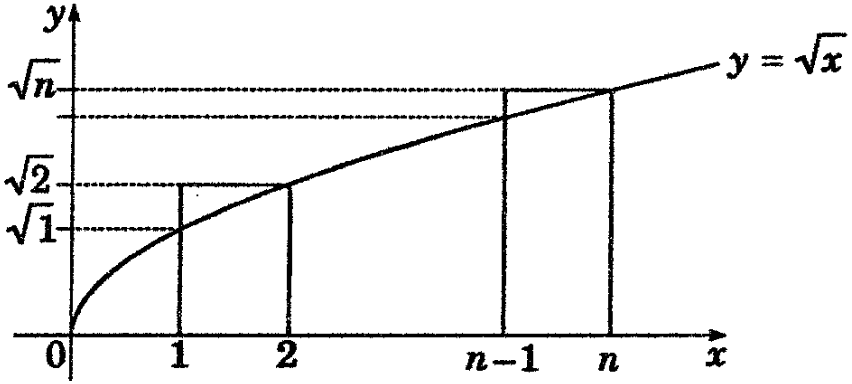
$$(x-2)(x+1) = 0$$

$$x = 2 \quad \text{or} \quad x = -1$$

$$\frac{x^2}{x+2} \leq 1$$

$$x < -2 \quad \text{or} \quad -1 \leq x \leq 2$$

(*ii*) (1990)



Consider the graph $y = \sqrt{x}$

a) Show that the graph is increasing for all $x \geq 0$

Curve is increasing when $\frac{dy}{dx} > 0$

$$y = \sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\therefore \frac{dy}{dx} > 0 \text{ for } x > 0$$

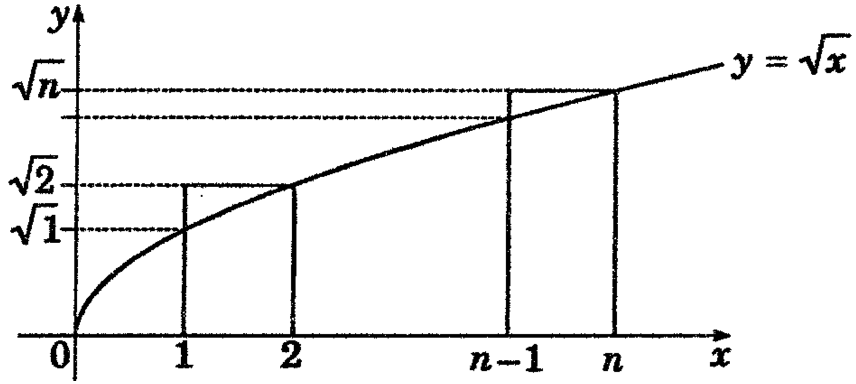
at $x = 0, y = 0$

when $x > 0, y > 0$

$\therefore$ curve is increasing for $x \geq 0$

b) Hence show that;

$$\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n} \geq \int_0^n \sqrt{x}dx = \frac{2}{3}n\sqrt{n}$$



As $\sqrt{x}$ is increasing;

Area outer rectangles ≥ Area under curve

$$\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n} \geq \int_0^n \sqrt{x}dx$$

$$= \left[\frac{2}{3}x\sqrt{x}\right]_0^n$$

$$= \frac{2}{3}n\sqrt{n}$$

$$\therefore \sqrt{1}+\sqrt{2}+\cdots+\sqrt{n} \geq \int_0^n \sqrt{x}dx = \frac{2}{3}n\sqrt{n}$$

c) Use mathematical induction to show that;

$$\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n} \le \frac{4n+3}{6}\sqrt{n} \text{ for all integers } n \ge 1$$

Prove the result is true for $n = 1$

$$L.H.S = \sqrt{1}$$
$$= 1$$

$$R.H.S = \frac{4(1)+3}{6}\sqrt{1}$$
$$= \frac{7}{6}$$

$$\therefore L.H.S \le R.H.S$$

Hence the result is true for $n = 1$

Assume the result is true for $n = k$, where $k \in \mathbb{Z}^+$

i.e. $\sqrt{1}+\sqrt{2}+\cdots+\sqrt{k} \le \frac{4k+3}{6}\sqrt{k}$

Prove the result is true for $n = k+1$

Prove $\sqrt{1} + \sqrt{2} + \ldots + \sqrt{k+1} - \frac{4k+7}{6}\sqrt{k+1} \le 0$

Proof:

$$\sqrt{1}+\sqrt{2}+\ldots+\sqrt{k+1}-\frac{4k+7}{6}\sqrt{k+1}=\sqrt{1}+\sqrt{2}+\ldots+\sqrt{k}-\frac{4k+1}{6}\sqrt{k+1}$$

$$\leq \frac{4k+3}{6}\sqrt{k}-\frac{4k+1}{6}\sqrt{k+1}$$

$$=\frac{\sqrt{(4k+3)^2k}-(4k+1)\sqrt{k+1}}{6}$$

$$=\frac{\sqrt{16k^3+24k^2+9k}-(4k+1)\sqrt{k+1}}{6}$$

$$=\frac{\sqrt{(k+1)(16k^2+8k+1)-1}-(4k+1)\sqrt{k+1}}{6}$$

$$<\frac{\sqrt{(k+1)(16k^2+8k+1)}-(4k+1)\sqrt{k+1}}{6}$$

$$=\frac{\sqrt{(k+1)(4k+1)^2}-(4k+1)\sqrt{k+1}}{6}$$

$$=\frac{(4k+1)\sqrt{k+1}-(4k+1)\sqrt{k+1}}{6}=0$$

$$\therefore \sqrt{1} + \sqrt{2} + \ldots + \sqrt{k+1} - \frac{4k+7}{6}\sqrt{k+1} \leq 0$$

Hence the result is true for $n = k+1$ if it is also true for $n = k$

Since the result is true for $n = 1$, then it is true $\forall\, n \in \mathbb{Z}^+$ by induction

d) Use b) and c) to estimate;

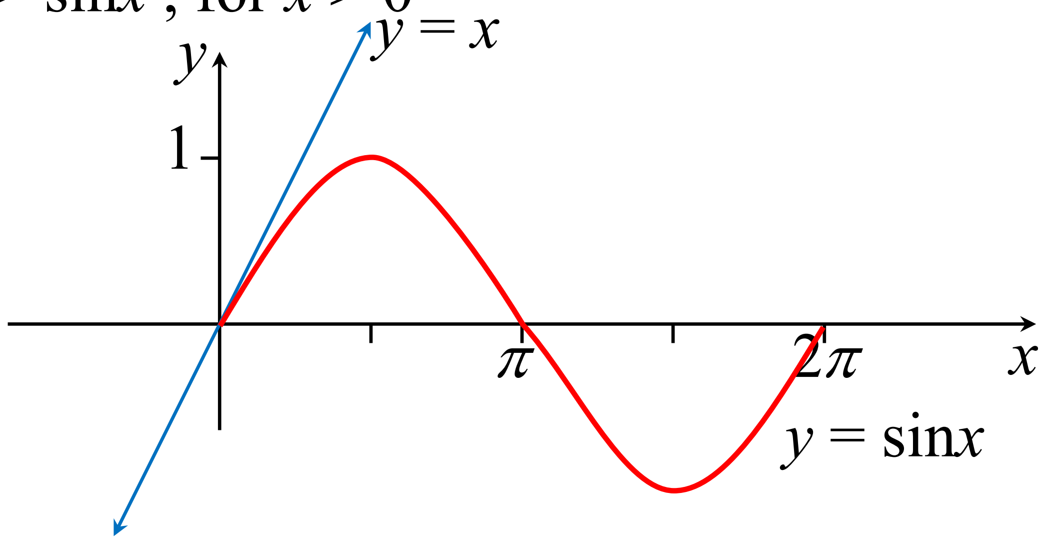
$\sqrt{1} + \sqrt{2} + \cdots + \sqrt{10000}$ to the nearest hundred

$$\frac{2}{3}n\sqrt{n} \leq \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} \leq \frac{4n+3}{6}\sqrt{n}$$

$$\frac{2}{3}(10000)\sqrt{10000} \leq \sqrt{1} + \sqrt{2} + \cdots + \sqrt{10000} \leq \frac{4(10000)+3}{6}\sqrt{10000}$$

$$666700 \leq \sqrt{1} + \sqrt{2} + \cdots + \sqrt{10000} \leq 666700$$

$\therefore \sqrt{1} + \sqrt{2} + \cdots + \sqrt{10000} = 666700$ to the nearest hundred (*iii*) Prove $x > \sin x$, for $x > 0$



$f(x) = x$ $\qquad$ $f(x) = \sin x$

$f'(x) = 1$ $\qquad$ $f'(x) = \cos x$

for $0 < x \le \frac{\pi}{2}$, $\cos x < 1$

$\therefore$ $y = x$ increases faster than $y = \sin x$

$x > \sin x$, for $0 < x \le \frac{\pi}{2}$

for $x > \frac{\pi}{2}$, $\sin x \le 1$

$\therefore x > \sin x$, for $x > \frac{\pi}{2}$

$\therefore x > \sin x$, for $x > 0$

(*iv*) 2023 Extension 2 HSC Question 16 b)

(i) Prove that $x > \ln x$, for $x > 0$

$$f(x) = x - \ln x$$

$$f'(x) = 1 - \frac{1}{x}$$

$$f''(x) = \frac{1}{x^2}$$

stationary points occur when $f'(x) = 0$

$$\text{i.e. } 1 - \frac{1}{x} = 0$$

$$x = 1$$

$$f''(1) = 1 > 0$$

$\therefore$ (1,1) is a minimum turning point

as there are no other stationary points, $f(x) \geq 1$ for it's natural domain

$$\text{i.e. } x - \ln x \geq 1 \quad \forall x > 0$$

$$x - \ln x > 0$$

$$x > \ln x$$

(ii) Using part (i), or otherwise, prove that for all positive integers n,

$$e^{n^2+n} > (n!)^2$$

$$n > \ln n \qquad \text{(from part (i))}$$

$$(n-1) > \ln(n-1)$$

$$(n-2) > \ln(n-2)$$

$$\vdots$$

$$1 > \ln(1)$$

Note: arithmetic series

$$n + (n-1) + (n-2) + \ldots + 1 > \ln n + \ln(n-1) + \ln(n-2) + \ldots + \ln(1)$$

$$\frac{n}{2}(n+1) > \ln[n(n-1)(n-2)\ldots(1)]$$

$$n(n+1) > 2\ln(n!)$$

$$n^2 + n > \ln\left[(n!)^2\right]$$

$$e^{n^2+n} > (n!)^2$$

($y = e^x$ is a continually increasing function)

Exercise 2F; 1, 3, 4, 6, 8, 9, 12, 14, 17