Pigeonhole Principle

If $n + 1$ or more objects are placed into n holes, then at least one hole must contain two, or more, objects

Let k be the maximum number of objects you can put in n holes, without having more than one object in a hole.
Prove $k < n + 1$

Proof

Assume $k \geq n+1$

Maximum number of objects = n i.e. each hole has one object

$\therefore k \leq n$ which is a contradiction

Thus if it cannot be true that $k \geq n+1$

$$\therefore \underline{k < n+1}$$

In general

If k, or more, objects are placed into n holes, then at least one hole must contain $\left\lceil \frac{k}{n} \right\rceil$, or more, objects

the ceiling function

if $\lceil x \rceil = y$,

then

$y - 1 < x \le y$

i.e. always round up

$\left\lceil \frac{k}{5} \right\rceil = 2$

$1 < \frac{k}{5} \le 2$

$5 < k \le 10$

e.g. (i) Integers are selected from

$\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

What is the least number of integers needed to ensure that there are a pair of numbers that add up to 15?

There are 5 possible pairs that will add up to 15

$\{3,12\}$ $\{4,11\}$ $\{5,10\}$ $\{6,9\}$ $\{7,8\}$

$n = 5$ (holes)

(objects)

The pigeonhole principle tells us that 6 integers need to be selected

(ii) Fifty points are placed in a 7 by 7 square. What is the maximum distance apart that the two closest points could be?

(holes)

The square could be divided up into 49, 1 by 1 squares (objects)

The pigeonhole principle tells us that if there are 50 points, there must be a square that contains at least two points

The closest distance between the points would be a maximum when there are only two points in one of these squares, and they are located at either end of the diagonal

$$d^2 = 1^2 + 1^2$$

$$d = \sqrt{2}$$

$\therefore$ the maximum distance that the closest pair of points could be is $\sqrt{2}$ units

(iii) Prove that in a group of n people, there will always be at least two people with the same number of friends in that group.

It is NOT possible for one person to have no friends and another person to have $(n - 1)$ friends, lets assume that nobody has no friends.

The possible number of friends a person could have is

$$\{1, 2, 3, \ldots, (n - 1)\}$$

i.e. $(n - 1)$ different number of friends a person could have

(holes)

By the pigeonhole principle there must be at least two people with the same number of friends

(iv) Sixteen positive integers are written on the board. At least how many numbers will leave the same remainder when divided by 5?

There are five possible remainders; 0 , 1 , 2 , 3 , 4

placing 16 objects into 5 holes

$$\left\lceil \frac{16}{5} \right\rceil = \lceil 3.2 \rceil$$

$\therefore$ at least 4 numbers will leave the same remainder

(v) A computer is generating random three letter words. How many words need to be generated to ensure that there is a word that has been generated eight times?

possible words $= 26^3 = 17576 = n$ (holes)

$$\left\lceil \frac{k}{17576} \right\rceil = 8$$

$$7 < \frac{k}{17576} \leq 8$$

$$123032 < k \leq 140608$$

$\therefore$ 123033 words need to be generated (*vi*) **2022 Extension 1 HSC Q12b**

A sports association manages 13 junior teams. It decides to check the age of all players. Any team that has more than 3 players above the age limit will be penalised.

A total of 41 players are found to be above the age limit.

Will any team be penalised? Justify your answer.

$$\left\lceil \frac{41}{13} \right\rceil = 4$$

by the pigeonhole principle at least one team will have four or more players above the age limit

So yes, a team will be penalised

Exercise 14H;
2, 5, 6, 7, 8, 9, 11, 12, 13,
15, 16, 17, 19, 21, 22, 23, 24