Reduction Formula

Reduction (or recurrence) formulae can be used when the integrand is raised to a power.

Integration by parts often used to find the formula.

e.g. (i) Find $\int_1^e x(\log x)^3\,dx$

$$u = (\log x)^n \qquad v = \frac{1}{2}x^2$$

$$I_n = \int_1^e x(\log x)^n\,dx \qquad du = \frac{n(\log x)^{n-1}}{x}dx \qquad dv = xdx$$

$$= \frac{1}{2}\left[x^2(\log x)^n\right]_1^e - \frac{n}{2}\int_1^e x(\log x)^{n-1}\,dx$$

$$= \frac{e^2}{2} - \frac{n}{2}\int_1^e x(\log x)^{n-1}\,dx$$

$$= \frac{e^2}{2} - \frac{n}{2}I_{n-1}$$

$$\therefore \int_1^e x(\log x)^3\,dx = I_3 = \frac{e^2}{2} - \frac{3}{2} I_2$$

$$= \frac{e^2}{2} - \frac{3}{2}\left(\frac{e^2}{2} - I_1\right)$$

$$= -\frac{e^2}{4} + \frac{3}{2} I_1$$

$$= -\frac{e^2}{4} + \frac{3}{2}\left(\frac{e^2}{2} - \frac{1}{2} I_0\right)$$

$$= \frac{e^2}{2} - \frac{3}{4} I_0$$

$$= \frac{e^2}{2} - \frac{3}{4}\int_1^e x\,dx$$

$$= \frac{e^2}{2} - \frac{3}{8}\left[x^2\right]_1^e$$

$$= \frac{e^2}{2} - \frac{3e^2}{8} + \frac{3}{8} \quad = \frac{e^2}{8} + \frac{3}{8}$$

OR

$$I_0 = \int_1^e x\,dx = \frac{1}{2}\left[x^2\right]_1^e$$

$$= \frac{1}{2}\left(e^2 - 1\right)$$

$$I_1 = \frac{e^2}{2} - \frac{1}{2} I_0 = \frac{e^2}{2} - \frac{1}{2} \times \frac{1}{2}\left(e^2 - 1\right)$$

$$= \frac{e^2}{4} + \frac{1}{4}$$

$$I_2 = \frac{e^2}{2} - I_1 = \frac{e^2}{2} - \frac{1}{4}\left(e^2 + 1\right)$$

$$= \frac{e^2}{4} - \frac{1}{4}$$

$$I_3 = \frac{e^2}{2} - \frac{3}{2} I_2 = \frac{e^2}{2} - \frac{3}{2} \times \frac{1}{4}\left(e^2 - 1\right)$$

$$= \frac{e^2}{8} + \frac{3}{8}$$

Integration by parts is the commonest way of getting reduction formulae, but it is not the only method.

Some just involve the use of a **trig identity**.

(ii) Given that $I_n = \int \cot^n x dx$, find I_6

$$I_n = \int \cot^n x dx$$

$$= \int \cot^{n-2} x \cot^2 x dx$$

$$= \int \cot^{n-2} x(\mathrm{cosec}^2 x - 1)dx$$

$$= \int \cot^{n-2} x\mathrm{cosec}^2 x dx - \int \cot^{n-2} x dx$$

$$= -\int u^{n-2} du - I_{n-2}$$

$$= -\frac{1}{n-1} u^{n-1} - I_{n-2}$$

$$= -\frac{1}{n-1} \cot^{n-1} x - I_{n-2}$$

$\int \cot^n x dx$

Uses the same technique, for all powers.

So just use that technique

$u = \cot x$

$du = -\mathrm{cosec}^2 x dx$

OR

Reduction formulae involving trig often reduces by 2. If you do not have to use parts you could try $I_n \pm I_{n+2}$

$$I_n + I_{n+2} = \int \cot^n x dx + \int \cot^{n+2} x dx$$

$$= \int \left(\cot^n x + \cot^{n+2} x\right) dx$$

$$= \int \left\{\cot^n x\left(1 + \cot^2 x\right)\right\} dx$$

$$= \int \cot^n x \operatorname{cosec}^2 x dx$$

$$= -\frac{\cot^{n+1} x}{n+1}$$

$$I_{n+2} = -\frac{\cot^{n+1} x}{n+1} - I_n$$

$$I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

$$\int \cot^6 x dx = I_6$$

$$= -\frac{1}{5}\cot^5 x - I_4$$

$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x + I_2$$

$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - I_0$$

$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - \int dx$$

$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - x + c$$

Sometimes integration by parts is not enough by itself.

Some involve the use of **parts** along with **algebraic manipulation** or use of a **trig identity**.

(iii) Given that $I_n = \int (1+x^2)^n \, dx$, find a reduction formula connecting I_n with I_{n-1}

$$u = (1+x^2)^n \qquad v = x$$

$$du = 2nx(1+x^2)^{n-1}\,dx \qquad dv = dx$$

$$I_n = \int (1+x^2)^n \, dx$$

$$= x(1+x^2)^n - 2n\int x^2(1+x^2)^{n-1}\,dx$$

$$= x(1+x^2)^n - 2n\int (1+x^2-1)(1+x^2)^{n-1}\,dx$$

$$= x(1+x^2)^n - 2n\int (1+x^2)(1+x^2)^{n-1}\,dx + 2n\int (1+x^2)^{n-1}\,dx$$

$$= x(1+x^2)^n - 2n\int (1+x^2)^{n}\,dx + 2n\int (1+x^2)^{n-1}\,dx$$

$$= x(1+x^2)^n - 2nI_n + 2nI_{n-1}$$

$$(2n+1)I_n = x(1+x^2)^n + 2nI_{n-1}$$

$$\therefore I_n = \frac{x(1+x^2)^n}{(2n+1)} + \frac{2n}{(2n+1)}I_{n-1}$$

e.g. (*iv*) (1987) Given that $I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$, prove that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

where n is an integer and $n \geq 2$, hence evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x dx$

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

$$u = \cos^{n-1} x \qquad v = \sin x$$

$$du = -(n-1)\cos^{n-2} x \sin x dx \qquad dv = \cos x dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x dx$$

$$= \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x dx$$

$$= \left\{\cos^{n-1}\frac{\pi}{2}\sin\frac{\pi}{2} - \cos^{n-1} 0 \sin 0\right\} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x\left(1-\cos^2 x\right)dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x dx - (n-1)\int_0^{\frac{\pi}{2}} \cos^n x dx$$

$$= (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore nI_n = (n-1)I_{n-2}$$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

OR

$$I_{n-2} - I_n$$

$$= \int_0^{\frac{\pi}{2}} \left(\cos^{n-2} x - \cos^n x\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-2} x\left(1 - \cos^2 x\right) dx$$

$$u = \sin x \qquad v = \frac{-\cos^{n-1} x}{n-1}$$

$$du = \cos x dx \qquad dv = \cos^{n-2} x \sin x dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x dx$$

$$= \frac{-1}{n-1}\left[\sin x \cos^{n-1} x\right]_0^{\frac{\pi}{2}} + \frac{1}{n-1}\int_0^{\frac{\pi}{2}} \cos^n dx$$

$$= \frac{1}{n-1} I_n$$

$$\therefore (n-1) I_{n-2} - (n-1) I_n = I_n$$

$$(n-1) I_{n-2} = n I_n$$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

$$\int_0^{\frac{\pi}{2}} \cos^5 x dx = I_5$$

$$= \frac{4}{5} I_3$$

$$= \frac{4}{5} \times \frac{2}{3} I_1$$

$$= \frac{8}{15} \int_0^{\frac{\pi}{2}} \cos x dx$$

$$= \frac{8}{15} [\sin x]_0^{\frac{\pi}{2}}$$

$$= \frac{8}{15} \left(\sin \frac{\pi}{2} - \sin 0 \right)$$

$$= \frac{8}{15}$$

(v) 2022 Extension 2 HSC Question 14 b)

Let $J_n = \int_0^1 x^n e^{-x} dx$, where n is a non-negative integer

(i) Show that $J_0 = 1 - \frac{1}{e}$

$$J_0 = \int_0^1 e^{-x} dx$$

$$= -\left[e^{-x}\right]_0^1$$

$$= -e^{-1} + e^0$$

$$= 1 - \frac{1}{e}$$

(ii) Show that $J_n \leq \dfrac{1}{n+1}$

$$0 \leq x \leq 1$$

$$\frac{1}{e} \leq e^{-x} \leq 1$$

(e^{-x} is continually decreasing)

Inequality signs are preserved when applying an increasing function and swapped when applying a decreasing function

$$J_n = \int_0^1 x^n e^{-x} dx \leq \int_0^1 x^n dx$$

$$= \frac{1}{n+1}\left[x^{n+1}\right]_0^1$$

$$= \frac{1}{n+1}(1-0)$$

$$= \frac{1}{n+1}$$

$$\therefore J_n \leq \frac{1}{n+1}$$

(iii) Show that $J_n = nJ_{n-1} - \frac{1}{e}$, for $n \geq 1$

$$J_n = \int_0^1 x^n e^{-x} dx \qquad u = x^n \qquad v = -e^{-x}$$

$$du = nx^{n-1}dx \qquad dv = e^{-x}dx$$

$$= \left[-x^n e^{-x}\right]_0^1 + n\int_0^1 x^{n-1}e^{-x}dx$$

$$= -e^{-1} + 0 + nJ_{n-1}$$

$$= nJ_{n-1} - \frac{1}{e}$$

(iv) Using parts (i) and (iii), show by mathematical induction, or otherwise that for all $n \geq 0$

$$J_n = n! - \frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!}$$

$$\text{Show } J_n = n! - \frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!}$$

Prove the result is true for $n = 0$

$$J_0 = 0! - \frac{0!}{e}\sum_{r=0}^{0}\frac{1}{r!}$$

$$= 1 - \frac{1}{e}$$

Hence the result is true for $n = 0$

Assume the result is true for $n = k$

$$\text{i.e } J_k = k! - \frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}$$

Prove the result is true for $n = k + 1$

$$\text{i.e Prove } J_{k+1} = (k+1)! - \frac{(k+1)!}{e}\sum_{r=0}^{k+1}\frac{1}{r!}$$

Proof:

$$J_{k+1} = (k+1)J_k - \frac{1}{e}$$

$$= (k+1)\left[k! - \frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}\right] - \frac{1}{e} \qquad \text{(by assumption)}$$

$$= (k+1)! - \frac{k+1!}{e}\sum_{r=0}^{k}\frac{1}{r!} - \frac{1}{e}$$

$$= (k+1)! - \frac{k+1!}{e}\sum_{r=0}^{k}\frac{1}{r!} - \frac{(k+1)!}{e} \times \frac{1}{(k+1)!}$$

$$= (k+1)! - \frac{(k+1)!}{e}\sum_{r=0}^{k+1}\frac{1}{r!}$$

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 0$, then it is true $\forall n \in \mathbb{Z} : n \geq 0$ by induction

(v) Using parts (ii) and (iv) prove that $e = \lim_{n \to \infty} \sum_{r=0}^{n} \frac{1}{r!}$

$$0 \le J_n \le \frac{1}{n+1}$$

$$0 \le n! - \frac{n!}{e} \sum_{r=0}^{n} \frac{1}{r!} \le \frac{1}{n+1}$$

$$0 \le e - \sum_{r=0}^{n} \frac{1}{r!} \le \frac{e}{(n+1)!}$$

$$0 \le \lim_{n \to \infty} \left(e - \sum_{r=0}^{n} \frac{1}{r!} \right) \le \lim_{n \to \infty} \frac{e}{(n+1)!}$$

$$0 \leq e - \lim_{n \to \infty} \sum_{r=0}^{n} \frac{1}{r!} \leq 0$$

$$\therefore e - \lim_{n \to \infty} \sum_{r=0}^{n} \frac{1}{r!} = 0$$

$$\lim_{n \to \infty} \sum_{r=0}^{n} \frac{1}{r!} = e$$

Exercise 4H;
2, 5, 6, 7, 8, 9, 10, 12, 14, 17