Differential Equations

Many of the basic laws of the physical, biological and social sciences are formulated in terms of mathematical relations involving certain known and unknown quantities and their derivatives.

e.g.

$$m\frac{d^2x}{dt^2} = F(x)$$

Newton's second law of motion
second order differential equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

damped oscillation
(shock absorber)
second order differential equation

$$\frac{dy}{dx} = -\frac{y(kx-b)}{x(cy-a)}$$

Lotka-Volterra equation
(predator-prey equation)
first order differential equation

Such relations are called **differential equations**

The **order** of a differential equation is the order of the highest order derivative appearing in the equation

Solutions to Differential Equations

Solving a differential equation, at a basic level, is equivalent to integration.

The **general solution** of a differential equation is a family of solution curves, similar to the indefinite integral

A **particular solution** of a differential equation is a particular curve from the family that solves an **initial value problem** i.e. it passes through a specific point. This would be similar to the definite integral.

e.g. $x\dfrac{dy}{dx} + y = 0$

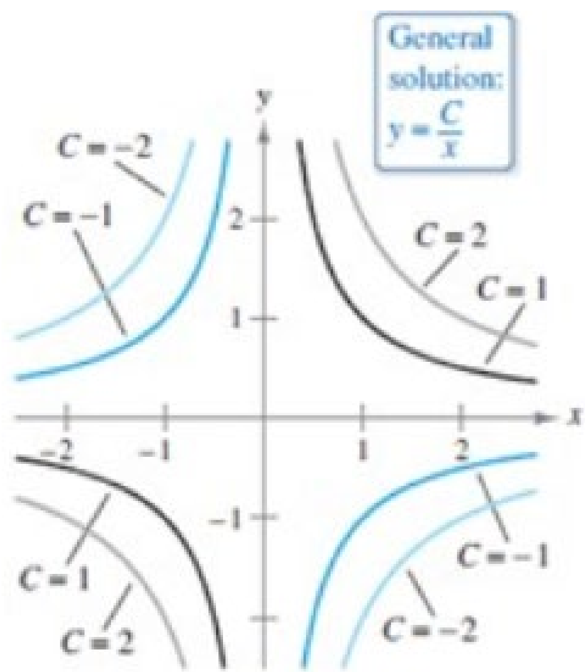
$$-\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$-\ln|y| = \ln|x| + k$$

$$\ln|xy| = -k$$

$$|xy| = e^{-k} = c$$

$$y = \frac{c}{x}$$



If $y(1) = 1$;

$$1 = \frac{c}{1}$$

$$c = 1$$

$\therefore$ particular solution is $y = \dfrac{1}{x}$

Linear Differential Equations

"**linear**" means;

$$\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

first order linear differential equation

$$\boxed{\frac{dy}{dx} + g(x)y = f(x)}$$

- if $f(x) = 0$; find the indefinite integral
- if $f(x) \neq 0$; multiply by an **integrating factor**; $u = e^{\int g(x)dx}$

e.g.(i) $y' + \frac{4y}{x} = 3x^2$

calculate integrating factor

$$\int \frac{4}{x}\,dx = 4\ln x$$

$$u = e^{\ln x^4}$$

$$= x^4$$

$$x^4 y' + 4x^3 y = 3x^6$$

you have created the "product rule" on the LHS

$$\frac{d}{dx}(x^4 y)$$

$$= (x^4)\left(\frac{dy}{dx}\right) + (y)(4x^3)$$

$$\frac{d}{dx}(x^4 y) = 3x^6$$

$$x^4 y = 3\int x^6\,dx$$

$$\underline{x^4 y = \frac{3}{7}x^7 + c}$$

$$\underline{y = \frac{3x^3}{7} + \frac{c}{x^4}}$$

(ii) a) Verify that $\frac{dy}{dx} = \frac{y+x}{x}$ is a first order linear differential equation

$$\frac{dy}{dx} = \frac{y+x}{x}$$

$$\frac{dy}{dx} = \frac{y}{x} + 1$$

$$\frac{dy}{dx} - \frac{y}{x} = 1$$ which is a first order linear differential equation

b) Find the general solution

$$\int \frac{-1}{x}\,dx = -\ln x$$

$$u = e^{-\ln x} = \frac{1}{x}$$

$$\frac{dy}{dx} - \frac{y}{x} = 1$$

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{1}{x}$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x}$$

integration factor $\times y$

$$\frac{y}{x} = \ln|x| + c$$

$$y = x\ln|x| + cx$$

(iii) $y' + 3y = x$

$$\int 3\,dx = 3x$$

$$u = e^{3x}$$

$$e^{3x}\frac{dy}{dx} + 3ye^{3x} = xe^{3x}$$

$$\frac{d}{dx}\left(ye^{3x}\right) = xe^{3x}$$

$$ye^{3x} = \int xe^{3x}\,dx$$

need to use the Extension 2 technique of integration by parts

$$ye^{3x} = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx$$

$$u = x \qquad v = \frac{1}{3}e^{3x}$$

$$du = dx \qquad dv = e^{3x}\,dx$$

$$ye^{3x} = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$$

$$\underline{y = \frac{x}{3} - \frac{1}{9} + ce^{-3x}}$$

(iv) **2022 Extension 1 HSC Q14 a)**

Find the particular solution to the differential equation $(x-2)\dfrac{dy}{dx}=xy$ that passes through the point $(0,1)$

$$(x-2)\frac{dy}{dx}=xy$$

$$\int_1^y \frac{dy}{y}=\int_0^x \frac{x\,dx}{x-2}$$

$$=\int_0^x \left(1+\frac{2}{x-2}\right)dx$$

$$\Big[\ln|y|\ \Big]_1^y=\Big[x+2\ln|x-2|\ \Big]_0^x$$

$$\ln|y|=x+2\ln|x-2|-0-2\ln 2$$

$$\ln|y| = \ln e^x + \ln\frac{(x-2)^2}{4}$$

$$= \ln\frac{(x-2)^2 e^x}{4}$$

$$|y| = \frac{(x-2)^2 e^x}{4}$$

$$y = \pm\frac{(x-2)^2 e^x}{4}$$

however we want the particular solution that passes through (0,1)

$$\therefore y = \frac{(x-2)^2 e^x}{4}$$

Exercise 13A;
1, 2, 4ac, 6ac, 7ace, 9, 10ace,
12a, 13, 14, 16, 17ac, 20