# Dot Product

A vector is a line segment in both 2D & 3D, the rules and properties remain the same in both dimensions.

If $\underset{\sim}{u} \neq 0 \wedge \underset{\sim}{v} \neq 0$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$$

NOTE: $\theta$ is acute or obtuse

If $\underset{\sim}{u} = 0 \vee \underset{\sim}{v} = 0$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$$

If $\underset{\sim}{u} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

## Dot Product Properties

(1) $-|\underset{\sim}{u}||\underset{\sim}{v}| \leq \underset{\sim}{u} \cdot \underset{\sim}{v} \leq |\underset{\sim}{u}||\underset{\sim}{v}|$

(2) $\underset{\sim}{u} \cdot \underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\sim}{u}$

(3) $\underset{\sim}{u} \cdot \underset{\sim}{u} = x_1{}^2 + y_1{}^2$
$= |\underset{\sim}{u}|^2$

(4) $\underset{\sim}{a} \cdot (\underset{\sim}{u} + \underset{\sim}{v}) = \underset{\sim}{a} \cdot \underset{\sim}{u} + \underset{\sim}{a} \cdot \underset{\sim}{v}$

(5) $(\underset{\sim}{u} + \underset{\sim}{v}) \cdot (\underset{\sim}{u} - \underset{\sim}{v}) = \underset{\sim}{u} \cdot \underset{\sim}{u} - \underset{\sim}{v} \cdot \underset{\sim}{v}$
$= |\underset{\sim}{u}|^2 - |\underset{\sim}{v}|^2$

(6) $\lambda \underset{\sim}{u} \cdot \underset{\sim}{v} = \lambda(\underset{\sim}{u} \cdot \underset{\sim}{v})$

(7) $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0 \Leftrightarrow \underset{\sim}{u} \perp \underset{\sim}{v}$

(8) $\underset{\sim}{u} \cdot \underset{\sim}{v} = \pm|\underset{\sim}{u}||\underset{\sim}{v}| \Leftrightarrow \underset{\sim}{u} \parallel \underset{\sim}{v}$

(9) $|\underset{\sim}{u}||\underset{\sim}{v}| > 0 \Rightarrow \underset{\sim}{u}$ and $\underset{\sim}{v}$ have the same direction
$|\underset{\sim}{u}||\underset{\sim}{v}| < 0 \Rightarrow \underset{\sim}{u}$ and $\underset{\sim}{v}$ have opposite directions

eg (i) Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be three 3-dimensional vectors.

Prove that $\underset{\sim}{a}\cdot(\underset{\sim}{b}+\underset{\sim}{c}) = \underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{c}$

$$\underset{\sim}{a}\cdot(\underset{\sim}{b}+\underset{\sim}{c})$$

$$= \left(a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}\right)\cdot\left[(b_1+c_1)\underset{\sim}{i} + (b_2+c_2)\underset{\sim}{j} + (b_3+c_3)\underset{\sim}{k}\right]$$

$$= a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3)$$

$$= a_1 b_1 + a_1 c_1 + a_2 b_2 + a_2 c_2 + a_3 b_3 + a_3 c_3$$

$$= (a_1 b_1 + a_2 b_2 + a_3 b_3) + (a_1 c_1 + a_2 c_2 + a_3 c_3)$$

$$= \underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{c}$$

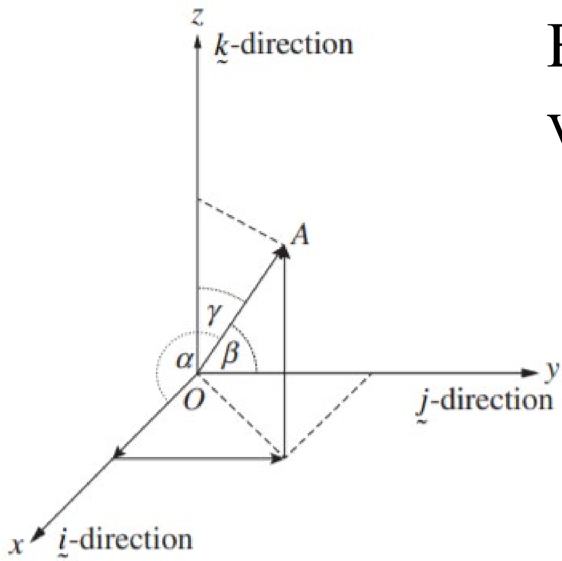
(ii) Prove that the vectors $3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$ and $-4\underset{\sim}{i} - 8\underset{\sim}{j} - \underset{\sim}{k}$ are perpendicular

$$\left(3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}\right)\cdot\left(-4\underset{\sim}{i} - 8\underset{\sim}{j} - \underset{\sim}{k}\right) = (3)(-4) + (-2)(-8) + (4)(-1)$$

$$= 0$$

$$\therefore \left(3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}\right) \perp \left(-4\underset{\sim}{i} - 8\underset{\sim}{j} - \underset{\sim}{k}\right)$$

(iii) The point $A$ has (non-zero) position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and the vector $\overrightarrow{OA}$ makes angles $\alpha$, $\beta$ and $\gamma$ with the $x$, $y$ and $z$ axes respectively




By taking a dot product with the three unit vectors $\underset{\sim}{i}$ , $\underset{\sim}{j}$ , $\underset{\sim}{k}$ prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

let $\underset{\sim}{u} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ ;

$\underset{\sim}{u} \cdot \underset{\sim}{i} = a$

also $\underset{\sim}{u} \cdot \underset{\sim}{i} = |\underset{\sim}{u}||\underset{\sim}{i}|\cos\alpha \; = \sqrt{a^2 + b^2 + c^2}\cos\alpha$

$\therefore \quad a = \sqrt{a^2 + b^2 + c^2}\cos\alpha$

$$\cos^2\alpha = \frac{a^2}{a^2 + b^2 + c^2}$$

Similarly; $\cos^2\beta = \dfrac{b^2}{a^2 + b^2 + c^2}$

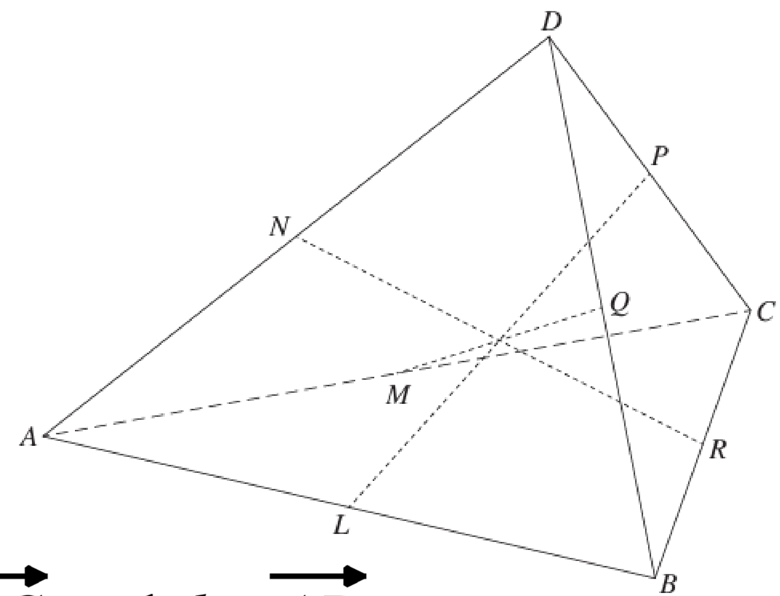
$$\cos^2\gamma = \frac{c^2}{a^2 + b^2 + c^2}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2}$$

$$\underline{=1}$$

$(iv)$ **2023 Extension 2 HSC Q15b)**

On the triangular pyramid $ABCD$, $L$ is the midpoint of $AB$, $M$ is the midpoint of $AC$, $N$ is the midpoint of $AD$, $P$ is the midpoint of $CD$, $Q$ is the midpoint of $BD$ and $R$ is the midpoint of $BC$.




Let $\underset{\sim}{b}=\overrightarrow{AB}$, $\underset{\sim}{c}=\overrightarrow{AC}$ and $\underset{\sim}{d}=\overrightarrow{AD}$

$(i)$ Show that $\overrightarrow{LP}=\frac{1}{2}(-\underset{\sim}{b}+\underset{\sim}{c}+\underset{\sim}{d})$

$$\overrightarrow{LP}=-\overrightarrow{AL}+\overrightarrow{AD}+\overrightarrow{DP}$$

$$=-\frac{1}{2}\overrightarrow{AB}+\overrightarrow{AD}+\frac{1}{2}\overrightarrow{DC}$$

$$\overrightarrow{LP}=-\frac{1}{2}\underset{\sim}{b}+\underset{\sim}{d}+\frac{1}{2}(\underset{\sim}{c}-\underset{\sim}{d})$$

$$=\frac{1}{2}(-\underset{\sim}{b}+\underset{\sim}{c}+\underset{\sim}{d})$$

$(ii)$ It can be shown that $\overrightarrow{MQ}=\frac{1}{2}(\underset{\sim}{b}-\underset{\sim}{c}+\underset{\sim}{d})$

and $\overrightarrow{NR}=\frac{1}{2}(\underset{\sim}{b}+\underset{\sim}{c}-\underset{\sim}{d})$

Prove that

$$\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{AC}\right|^2+\left|\overrightarrow{AD}\right|^2+\left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{BD}\right|^2+\left|\overrightarrow{CD}\right|^2=4\left(\left|\overrightarrow{LP}\right|^2+\left|\overrightarrow{MQ}\right|^2+\left|\overrightarrow{NR}\right|^2\right)$$

$$\left|\overrightarrow{LP}\right|^2=\overrightarrow{LP}\cdot\overrightarrow{LP}$$

$$=\frac{1}{4}(-\underset{\sim}{b}+\underset{\sim}{c}+\underset{\sim}{d})\cdot(-\underset{\sim}{b}+\underset{\sim}{c}+\underset{\sim}{d})$$

$$=\frac{1}{4}\left(\underset{\sim}{b}\cdot\underset{\sim}{b}+\underset{\sim}{c}\cdot\underset{\sim}{c}+\underset{\sim}{d}\cdot\underset{\sim}{d}+2(-\underset{\sim}{b}\cdot\underset{\sim}{c}-\underset{\sim}{b}\cdot\underset{\sim}{d}+\underset{\sim}{c}\cdot\underset{\sim}{d})\right)$$

$$=\frac{1}{4}\left(|\underset{\sim}{b}|^2+|\underset{\sim}{c}|^2+|\underset{\sim}{d}|^2+2(-\underset{\sim}{b}\cdot\underset{\sim}{c}-\underset{\sim}{b}\cdot\underset{\sim}{d}+\underset{\sim}{c}\cdot\underset{\sim}{d})\right)$$

similarly

$$\left|\overrightarrow{MQ}\right|^2=\frac{1}{4}\left(|\underset{\sim}{b}|^2+|\underset{\sim}{c}|^2+|\underset{\sim}{d}|^2+2(-\underset{\sim}{b}\cdot\underset{\sim}{c}+\underset{\sim}{b}\cdot\underset{\sim}{d}-\underset{\sim}{c}\cdot\underset{\sim}{d})\right)$$

$$\left|\overrightarrow{NR}\right|^2=\frac{1}{4}\left(|\underset{\sim}{b}|^2+|\underset{\sim}{c}|^2+|\underset{\sim}{d}|^2+2(\underset{\sim}{b}\cdot\underset{\sim}{c}-\underset{\sim}{b}\cdot\underset{\sim}{d}-\underset{\sim}{c}\cdot\underset{\sim}{d})\right)$$

$$4\left( \left|\overrightarrow{LP}\right|^2 + \left|\overrightarrow{MQ}\right|^2 + \left|\overrightarrow{NR}\right|^2 \right) = 3|\underset{\sim}{b}|^2 + 3|\underset{\sim}{c}|^2 + 3|\underset{\sim}{d}|^2 - 2(\underset{\sim}{b}\cdot\underset{\sim}{c} + \underset{\sim}{b}\cdot\underset{\sim}{d} + \underset{\sim}{c}\cdot\underset{\sim}{d})$$

$$\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{AD}\right|^2 + \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{BD}\right|^2 + \left|\overrightarrow{CD}\right|^2$$

$$= |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 + |\underset{\sim}{d}|^2 + |\underset{\sim}{c} - \underset{\sim}{b}|^2 + |\underset{\sim}{d} - \underset{\sim}{b}|^2 + |\underset{\sim}{d} - \underset{\sim}{c}|^2$$

$$= |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 + |\underset{\sim}{d}|^2 + (\underset{\sim}{c} - \underset{\sim}{b})\cdot(\underset{\sim}{c} - \underset{\sim}{b}) + (\underset{\sim}{d} - \underset{\sim}{b})\cdot(\underset{\sim}{d} - \underset{\sim}{b}) + (\underset{\sim}{d} - \underset{\sim}{c})\cdot(\underset{\sim}{d} - \underset{\sim}{c})$$

$$= |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 + |\underset{\sim}{d}|^2 + |\underset{\sim}{c}|^2 - 2\underset{\sim}{b}\cdot\underset{\sim}{c} + |\underset{\sim}{b}|^2 + |\underset{\sim}{d}|^2 - 2\underset{\sim}{b}\cdot\underset{\sim}{d} + |\underset{\sim}{b}|^2 + |\underset{\sim}{d}|^2 - 2\underset{\sim}{c}\cdot\underset{\sim}{d} + |\underset{\sim}{c}|^2$$

$$= 3|\underset{\sim}{b}|^2 + 3|\underset{\sim}{c}|^2 + 3|\underset{\sim}{d}|^2 - 2(\underset{\sim}{b}\cdot\underset{\sim}{c} + \underset{\sim}{b}\cdot\underset{\sim}{d} + \underset{\sim}{c}\cdot\underset{\sim}{d})$$

$$= 4\left( \left|\overrightarrow{LP}\right|^2 + \left|\overrightarrow{MQ}\right|^2 + \left|\overrightarrow{NR}\right|^2 \right)$$

**Exercise 5C; 1a, 4, 6, 9, 11, 14b, 15b, 16a, 18, 20, 21**