Separation of Variables

A first order differential equation is **separable** if it can be written in the form;

$$\frac{dy}{dx} = f(x)g(y)$$

which can then be rewritten as;

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

Note: "trivial" solutions of $g(y) = 0$, may get lost in this method due to "division by zero" issues.

Always perform a quick mental check for possible "trivial" solutions e.g. (*i*) $\frac{dy}{dx} = x^2(1 + y^2)$

$$\int \frac{dy}{1 + y^2} = \int x^2 dx$$

$$\tan^{-1} y = \frac{x^3}{3} + c$$

$$y = \tan\left(\frac{x^3}{3} + c\right)$$

(ii) $\frac{dy}{dx} = x^2 y$

$$\int \frac{dy}{y} = \int x^2 dx$$

$y = 0$ is a "trivial" solution

$$\ln|y| = \frac{x^3}{3} + c$$

$$|y| = e^{\frac{x^3}{3} + c}$$

$$|y| = Ce^{\frac{x^3}{3}} \qquad (C = e^c > 0)$$

$$y = Ae^{\frac{x^3}{3}}$$

$$\therefore y = 0 \text{ or } y = Ae^{\frac{x^3}{3}}$$

(iii) $\frac{dy}{dx} = \left(\frac{\cos y}{x}\right)^2 \; ; \; y(1) = \frac{\pi}{4}$

$$\int_{\frac{\pi}{4}}^{y} \sec^2 y \, dy = \int_1^x \frac{dx}{x^2}$$

$$\Big[\tan y\Big]_{\frac{\pi}{4}}^{y} = \left[-\frac{1}{x}\right]_1^x$$

$$\tan y - 1 = -\frac{1}{x} + 1$$

$$\tan y = 2 - \frac{1}{x}$$

$$y = \tan^{-1}\left(2 - \frac{1}{x}\right)$$

Use definite integrals when solving initial value problems

(iv) $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2x^2}$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{y}{x}\right)^2 \qquad \text{let } u = \frac{y}{x} \Rightarrow y = ux$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = \frac{1}{2} + \frac{1}{2}u^2$$

$$x\frac{du}{dx} = \frac{1}{2} - u + \frac{1}{2}u^2$$

$$= \frac{1 - 2u + u^2}{2}$$

$$= \frac{(1 - u)^2}{2}$$

$$2\int \frac{du}{(1 - u)^2} = \int \frac{dx}{x}$$

$$\frac{2}{1 - u} = \ln|x| + c$$

$$1 - u = \frac{2}{\ln|x| + c}$$

$$u = 1 - \frac{2}{\ln|x| + c}$$

$$\frac{y}{x} = 1 - \frac{2}{\ln|x| + c}$$

$$y = x - \frac{2x}{\ln|x| + c}$$

Not separable?

try rewriting as

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Exercise 13C; 2, 3, 5bd, 6, 8bdf, 9, 12bcf, 13, 14, 15, 16, 17, 18