Variance

Let X be a **discrete random variable**, then the variance of X is;

$$\text{Var}(X) = \sum (x-\mu)^2 \, p(x)$$

where $p(x) = \text{P}(X = x) \geq 0$

$$\mu = \text{E}(X)$$

Note: $x - \mu$ is the **deviation** of x from μ

Var(X) is a measure of spread

e.g. Using the data of the milk marketing survey, find the variance of the number of litres of milk consumed in a week by a family.

x	**0**	**1**	**2**	**3**	**4**	**5**	**Σ**
$p(x)$	$\frac{2}{25}$	$\frac{5}{25}$	$\frac{9}{25}$	$\frac{5}{25}$	$\frac{3}{25}$	$\frac{1}{25}$	1
$xp(x)$	0	$\frac{5}{25}$	$\frac{18}{25}$	$\frac{15}{25}$	$\frac{12}{25}$	$\frac{5}{25}$	2.2 ← μ

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$(x-\mu)^2$	$\frac{121}{25}$	$\frac{36}{25}$	$\frac{1}{25}$	$\frac{16}{25}$	$\frac{81}{25}$	$\frac{196}{25}$	—

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$(x-\mu)^2$	$\frac{121}{25}$	$\frac{36}{25}$	$\frac{1}{25}$	$\frac{16}{25}$	$\frac{81}{25}$	$\frac{196}{25}$	—
$(x-\mu)^2 p(x)$	$\frac{242}{625}$	$\frac{180}{625}$	$\frac{9}{625}$	$\frac{80}{625}$	$\frac{243}{625}$	$\frac{196}{625}$	1.52

μ

Var(X)

$= \sum (x-\mu)^2 p(x)$

$= \underline{1.52}$

alternatively

$$\mathrm{Var}(X) = \sum (x-\mu)^2 \, p(x)$$
$$= \mathrm{E}\left[(X-\mu)^2\right]$$
$$= \mathrm{E}\left(X^2 - 2\mu X + \mu^2\right)$$
$$= \mathrm{E}\left(X^2\right) - 2\mu \mathrm{E}(X) + \mu^2$$
$$= \mathrm{E}\left(X^2\right) - 2\mu \times \mu + \mu^2$$

$$\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \mu^2$$

where $\mathrm{E}\left(X^2\right) = \sum x^2 p(x)$

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$p(x)$	$\frac{2}{25}$	$\frac{5}{25}$	$\frac{9}{25}$	$\frac{5}{25}$	$\frac{3}{25}$	$\frac{1}{25}$	1
$xp(x)$	0	$\frac{5}{25}$	$\frac{18}{25}$	$\frac{15}{25}$	$\frac{12}{25}$	$\frac{5}{25}$	2.2

μ

alternatively

$$\operatorname{Var}(X) = \sum (x-\mu)^2\, p(x)$$
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$$= \mathrm{E}\left(X^2 - 2\mu X + \mu^2\right)$$
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$$\operatorname{Var}(X) = \mathrm{E}\left(X^2\right) - \mu^2$$
$$\text{where } \mathrm{E}\left(X^2\right) = \sum x^2 p(x)$$

x	**0**	**1**	**2**	**3**	**4**	**5**	**Σ**
$p(x)$	$\frac{2}{25}$	$\frac{5}{25}$	$\frac{9}{25}$	$\frac{5}{25}$	$\frac{3}{25}$	$\frac{1}{25}$	1
$xp(x)$	0	$\frac{5}{25}$	$\frac{18}{25}$	$\frac{15}{25}$	$\frac{12}{25}$	$\frac{5}{25}$	2.2 ← μ
$x^2p(x)$	0	$\frac{5}{25}$	$\frac{36}{25}$	$\frac{45}{25}$	$\frac{48}{25}$	$\frac{25}{25}$	6.36 ← $\mathrm{E}(X^2)$

$$\operatorname{Var}(X) = \mathrm{E}\left(X^2\right) - \mu^2$$
$$= 6.36 - (2.2)^2 = \underline{1.52}$$

Standard Deviation (σ)

If the random variable, X, is measured in units
then variance $(\mathrm{E}(X^2) - \mu^2)$ is measured in units^2.

Standard deviation is used to measure spread using the same units as the random variable.

$$\sigma = \sqrt{\mathrm{Var}(X)}$$

or

$$\sigma^2 = \mathrm{Var}(X)$$

e.g. $\sigma = \sqrt{1.52}$

$= 1.23288...$

$= \underline{1.23}$ (to 2 dp)

Standardising Data

Data standardisation is the process of making sure your data set can be compared to other data sets

Original data

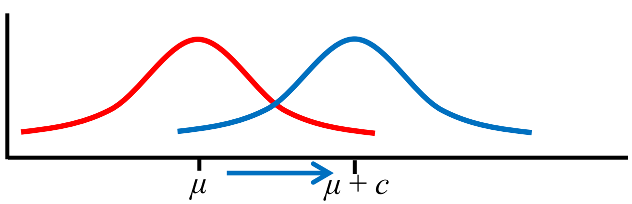
$$\mathrm{E}(X)=\mu \quad \text{and} \quad \mathrm{Var}(X)=\sigma^2$$

adding a constant amount to each piece of data

$$Y=X+c$$

$$\begin{aligned}\mathrm{E}(Y)&=\mathrm{E}(X+c)\\&=\mathrm{E}(X)+\mathrm{E}(c)\\&=\mu+c\end{aligned}$$

simply adding a constant amount does not change the shape of the data, it only translates it c units



$$\begin{aligned}\mathrm{Var}(Y)&=\mathrm{Var}(X+c)\\&=\mathrm{E}\left[(X+c)^2\right]-(\mu+c)^2\\&=\mathrm{E}(X^2+2cX+c^2)-(\mu^2+2c\mu+c^2)\\&=\mathrm{E}(X^2)+2c\mathrm{E}(X)+c^2-\mu^2-2c\mu-c^2\\&=\mathrm{E}(X^2)+2c\mu+c^2-\mu^2-2c\mu-c^2\\&=\mathrm{E}(X^2)-\mu^2\\&=\mathrm{Var}(X)=\sigma^2\end{aligned}$$

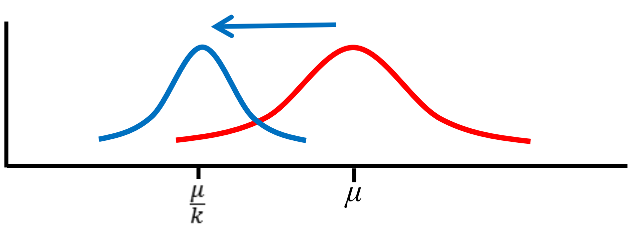
dividing each piece of data by a constant amount

$$Z = \frac{X}{k}$$

$$\mathrm{E}(Z) = \mathrm{E}\left(\frac{X}{k}\right) = \frac{\mathrm{E}(X)}{k} = \frac{\mu}{k}$$

$$\mathrm{Var}(Z) = \mathrm{Var}\left(\frac{X}{k}\right) = \mathrm{E}\left[\left(\frac{X}{k}\right)^2\right] - \left(\frac{\mu}{k}\right)^2 = \frac{\mathrm{E}(X^2)}{k^2} - \frac{\mu^2}{k^2} = \frac{\mathrm{E}(X^2) - \mu^2}{k^2} = \frac{\mathrm{Var}(X)}{k^2} = \frac{\sigma^2}{k^2} = \left(\frac{\sigma}{k}\right)^2$$

dividing by a constant amount changes the shape of the data, as well as altering the mean



z-scores

A common standardisation is to "normalise" the data by creating a z-score

$$z = \frac{x-\mu}{\sigma}$$

$$\mathrm{E}(Z) = \mathrm{E}\left(\frac{X-\mu}{\sigma}\right)$$

$$= \frac{\mathrm{E}(X-\mu)}{\sigma}$$

$$= \frac{\mu-\mu}{\sigma}$$

$$= 0$$

$$\mathrm{Var}(Z) = \mathrm{Var}\left(\frac{X-\mu}{\sigma}\right)$$

$$= \frac{\mathrm{Var}(X-\mu)}{\sigma^2}$$

$$= \frac{\sigma^2}{\sigma^2} = 1$$

$$\therefore \mu_Z = 0 \quad \text{and} \quad \sigma_Z = 1$$

Exercise 13C; 1, 2, 3bd, 4, 6, 8, 9, 10, 11, 12abc, 13