Differential Equations Of The Form $y' = g(y)$

Differential equations of the form;

$$\frac{dy}{dx} = g(y)$$

are easily **separable** and written in the form;

$$\int dx = \int \frac{dy}{g(y)}$$

Note: they can also be considered as a first order linear DE

$$\frac{dy}{dx} - g(y) = f(x) \text{ , where } f(x) = 0$$

e.g. $(i)\ \frac{dy}{dx} = k(y - N)\ ;\ y(0) = N + A$

$$\int_{N+A}^{y} \frac{dy}{y - N} = k\int_{0}^{x} dx$$

$$\Big[\ln|y - N|\Big]_{N+A}^{y} = kx$$

$$kx = \ln\left|\frac{y - N}{A}\right|$$

$$e^{kx} = \left|\frac{y - N}{A}\right|$$

$$y - N = Ae^{kx}$$

$$\underline{y = N + Ae^{kx}}$$

this is our modified growth & decay equation

$$\frac{dP}{dt} = k(P - N)$$

$$P = N + Ae^{kt}$$

(ii) $\frac{dy}{dx} = e^{2y-1}$

$$\int e^{1-2y}\,dy = \int dx$$

$$-\frac{1}{2}e^{1-2y} = x + c$$

$$e^{1-2y} = -2x + c$$

$$1 - 2y = \ln(-2x + c)$$

$$2y = 1 - \ln(c - 2x)$$

$$y = \frac{1}{2}[1 - \ln(c - 2x)]$$

(iii) $\frac{dy}{dx} = \sqrt{1 - y^2}$

$$\int \frac{dy}{\sqrt{1 - y^2}} = \int dx$$

$$\sin^{-1} y = x + c$$

$$y = \sin(x + c)$$

The Logistic Equation

The standard logistic equation is the solution of the first order differential equation

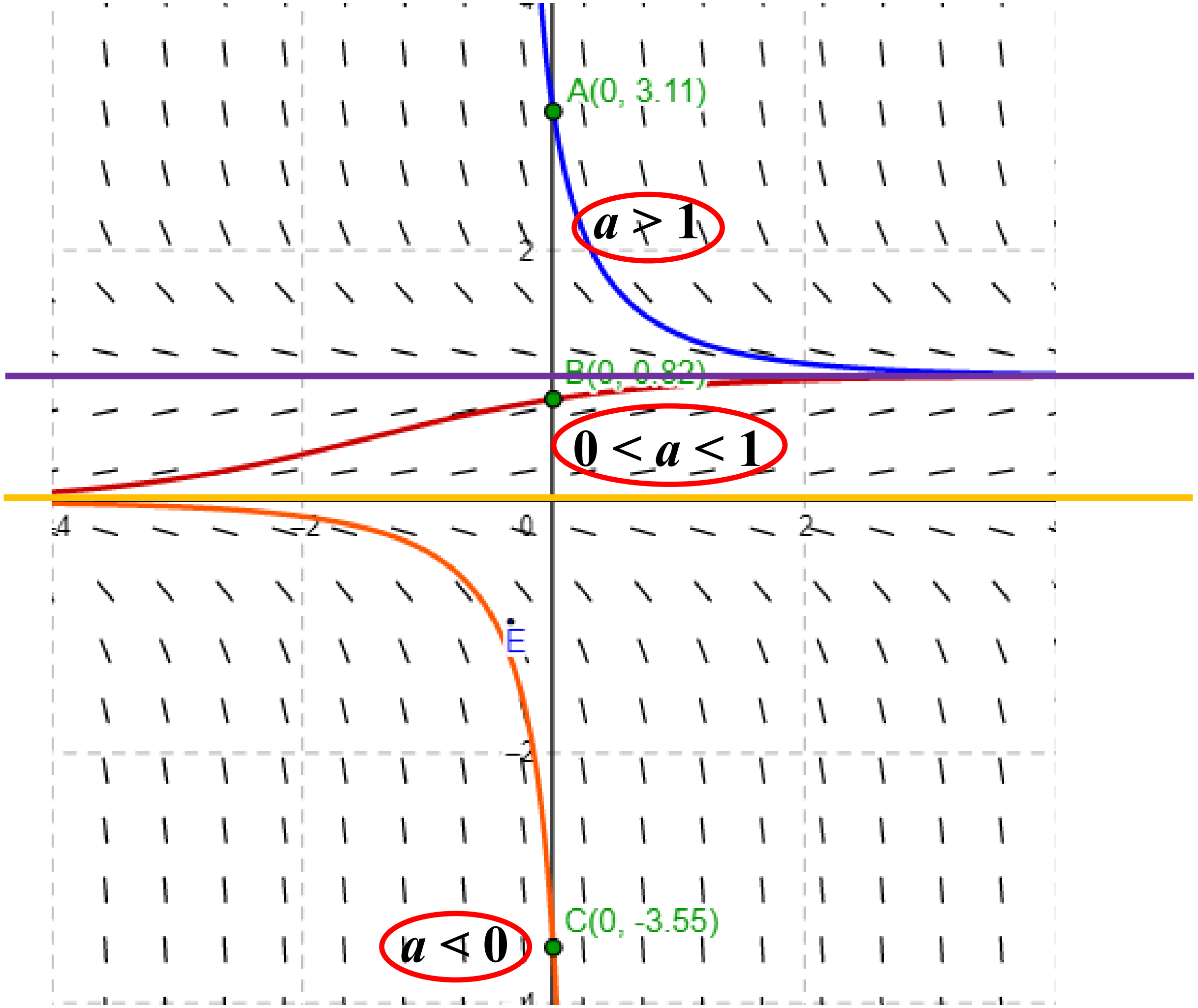
$$\frac{d}{dx}(f(x)) = f(x)(1 - f(x))$$

In this course we will restrict the logistic equation to ones of the form

$$\frac{dy}{dx} = ky(P - y)$$

e.g. $(i)\ \frac{dy}{dx} = y(1-y)\ ;\ y(0) = a$

If we look at the slope field, we can see that there are three basic curves, depending upon the value of a



as well as two trivial solutions when $a = 0$ and $a = 1$

$$\frac{dy}{dx} = y(1-y)$$

$$\int_a^y \frac{dy}{y(1-y)} = \int_0^x dx$$

$$x = \int_a^y \left(\frac{1}{y} + \frac{1}{1-y}\right) dy$$

$$= \left[\ln \left| \frac{y}{1-y} \right| \right]_a^y$$

$$= \ln \left| \frac{y(1-a)}{a(1-y)} \right|$$

$$e^x = \left| \frac{y(1-a)}{a(1-y)} \right|$$

$$Ae^x = \frac{y}{1-y} \qquad \left(|A| = \frac{a}{1-a} \right)$$

$$Ae^x(1-y) = y$$

$$y(1 + Ae^x) = Ae^x$$

$$y = \frac{Ae^x}{1 + Ae^x}$$

$$y = \frac{1}{Be^{-x} + 1} \qquad \left(B = \frac{1}{A} \right)$$

$$\therefore y = 0 \text{ , } y = 1 \text{ or } y = \frac{1}{Be^{-x} + 1}$$

(ii) Find any inflection points in the logistic curve

possible inflection points occur when $\frac{d^2y}{dx^2} = 0$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\{y(1-y)\}$$

$$= \frac{d}{dy}\{y(1-y)\} \times \frac{dy}{dx}$$

$$= \{-y + (1-y)\}y(1-y)$$

$$= y(1-2y)(1-y)$$

$$y = 0\ ,\ \frac{1}{2} \text{ or } 1$$

$\therefore$ the only possible inflection point is when $y = \frac{1}{2}$

$$Be^{-x} = 1$$

$$e^{x} = B$$

$$x = \ln B$$

$$= \ln\left|\frac{1-a}{a}\right|$$

Exercise 13D; 1, 3b, 5c, 6bdf, 8, 9, 11, 12, 14, 16, 17, 18, 20, 21

the slope field shows that there is a change in concavity

thus $\left(\ln\left|\frac{1-a}{a}\right|, \frac{1}{2}\right)$ is the point of inflection