Differential Equations In the Real World

e.g.

(i) A tank contains 30 litres of a solution of a chemical in water.

The concentration of the chemical is reduced by running pure water into the tank at a rate 1 litre per minute and allowing the solution to run out of the tank at a rate of 2 litres per minute.

The tank contains x litres of the chemical at time t minutes after the dilution starts.

a) Show that $\frac{dx}{dt} = -\frac{2x}{30 - t}$

Every minute the volume of the tank reduces by 1 litre, thus in t minutes the volume will be $(30 - t)$ litres

At time t, the fraction of the solution that is chemical is

$$\frac{x}{30 - t}$$

The solution is escaping at 2 litres per minute, so the rate of flow of chemical out of the tank is

$$\frac{dx}{dt} = -\frac{2x}{30 - t}$$

b) Find the fraction of the original chemical still in the tank after 20 minutes

Let the original volume of chemical be V

$$\frac{dx}{dt} = -\frac{2x}{30-t}$$

$$\int_V^x \frac{dx}{2x} = \int_0^t \frac{-dt}{30-t}$$

$$\frac{1}{2}\Big[\ln|x|\Big]_V^x = \Big[\ln|30-t|\Big]_0^t$$

$$\ln\left|\frac{x}{V}\right| = 2\ln\left|\frac{30-t}{30}\right|$$

$$\frac{x}{V} = \frac{(30-t)^2}{900}$$

$$x = \frac{V(30-t)^2}{900}$$

when $t = 20$; $x = \frac{V(30-20)^2}{900}$

$$= \frac{V}{9}$$

$\therefore$ after 20 minutes $\frac{1}{9}$ of the original chemical is left

(ii) The population of foxes on an island is modelled by the logistic equation $\frac{dy}{dt} = y(1 - y)$, where y is the fraction of the island's carrying capacity after t years.

At time $t = 0$, the population of foxes is estimated to be one-quarter of the island's carrying capacity.

a) Use the substitution $y = \frac{1}{1 - w}$ to transform the logistic equation to

$$\frac{dw}{dt} = -w$$

$$y = \frac{1}{1 - w}$$

$$\frac{dy}{dt} = \frac{1}{(1 - w)^2} \times \frac{dw}{dt}$$

$$\frac{dy}{dt} = y(1 - y)$$

$$\frac{1}{(1 - w)^2}\frac{dw}{dt} = \frac{1}{1 - w}\left(1 - \frac{1}{1 - w}\right)$$

$$\frac{dw}{dt} = \frac{1}{1 - w}\left(-\frac{w}{1 - w}\right)(1 - w)^2$$

$$= -w$$

b) Using the solution of $\frac{dw}{dt} = -w$ find the solution of the logistic equation for y satisfying the initial conditions.

$$\frac{dw}{dt} = -w \qquad \text{when } t = 0,\ y = \frac{1}{4};$$

$$\frac{1}{4} = \frac{1}{1 - w}$$

$$w = -3$$

$$w = -3e^{-t}$$

$$y = \frac{1}{1 + 3e^{-t}}$$

c) How long will it take for the fox population to reach three-quarters of the island's carrying capacity?

$$\frac{3}{4} = \frac{1}{1 + 3e^{-t}}$$

$$1 + 3e^{-t} = \frac{4}{3}$$

$$3e^{-t} = \frac{1}{3}$$

$$e^{t} = 9$$

$$t = \ln 9 = 2.19722\ldots$$

$\therefore$ It will take 2.2 years for the population to reach three-quarters capacity

(iii) A rumour is spreading amongst the 1000 students in a school at a rate proportional to those at have heard it, x, and those who have not heard it, $1000 - x$. As time progresses people care less about the rumour, i.e. the proportion rate is not constant.

The rate the rumour spreads is modelled by the logistic equation

$$\frac{dx}{dt} = \frac{Kx(1000 - x)}{t + 1}$$

If initially one student starts spreading the rumour, and after 1 hour 50 students have already heard the rumour, how many students have heard the rumour after 3 hours?

$$\frac{dx}{dt} = \frac{Kx(1000 - x)}{t + 1}$$

$$\int_1^{50} \frac{dx}{x(1000 - x)} = K\int_0^1 \frac{dt}{1 + t}$$

$$\frac{1}{1000}\int_1^{50} \left(\frac{1}{x} + \frac{1}{1000 - x}\right)dx = K[\ln(1 + t)]_0^1$$

$$\left[\ln\left(\frac{x}{1000-x}\right)\right]_1^{50} = 1000K\ln 2$$

$$\ln\left(\frac{1}{19}\times\frac{999}{1}\right) = 1000K\ln 2$$

$$K = \frac{\ln\left(\frac{999}{19}\right)}{1000\ln 2}$$

$$\int_1^x \frac{dx}{x(1000-x)} = K\int_0^3 \frac{dt}{1+t}$$

$$\left[\ln\left(\frac{x}{1000-x}\right)\right]_1^{x} = 1000K\Big[\ln(1+t)\Big]_0^3$$

$$\ln\left(\frac{999x}{1000-x}\right) = C$$

$$C = 1000\times\frac{\ln\left(\frac{999}{19}\right)}{1000\ln 2}\times(\ln 4)$$

$$= 2\ln\left(\frac{999}{19}\right)$$

$$\frac{999x}{1000 - x} = e^C$$

$$999x = e^C(1000 - x)$$

$$x(999 + e^C) = 1000e^C$$

$$x = \frac{1000e^C}{999 + e^C}$$

$$= 734.558\ldots$$

$\therefore$ 734 students have heard the rumour after 3 hours

Exercise 13E; 2, 4, 5, 6, 7, 11, 12, 13, 15, 16, 20