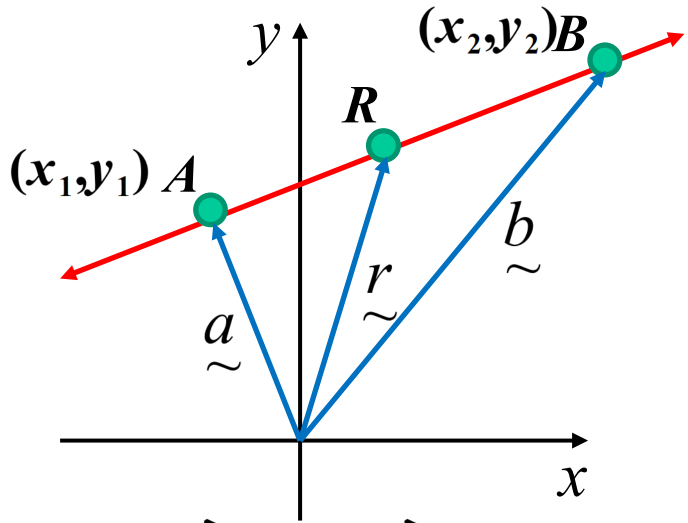
Vector Equation of a Line in 2D



If R is on the line AB then $\overrightarrow{AR} = \lambda \overrightarrow{AB}$, $\lambda \in \mathbb{R}$

$$\underset{\sim}{r} - \underset{\sim}{a} = \lambda(\underset{\sim}{b} - \underset{\sim}{a})$$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a})$$ **vector equation of a line**

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \lambda \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

$$x = x_1 + \lambda(x_2 - x_1)$$
$$y = y_1 + \lambda(y_2 - y_1)$$

parametric equation of a line

$$\therefore \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \lambda$$

cartesian equation of a line

$$(y_2 - y_1)(x - x_1) = (x_2 - x_1)(y - y_1)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - y_1 = m(x - x_1)$$

e.g. Write a vector equation of the line passing through (3,–5) and (–2,–8)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

OR $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -8 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$NOT\dot{E}$
cartsian equation
$$y = \frac{3}{5}x - \frac{34}{3}$$

notice the similarity to the slope

Recall from complex numbers, rotation of a vector 90° is multiplication by i

$$\begin{pmatrix} run \\ rise \end{pmatrix} \times i = (run + irise)(i)$$
$$= (irun - rise)$$
$$= \begin{pmatrix} -rise \\ run \end{pmatrix}$$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} \perp \begin{pmatrix} -b \\ a \end{pmatrix}$$

and using the dot product

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} -b \\ a \end{pmatrix} = -ab + ab$$
$$= 0$$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} \perp \begin{pmatrix} -b \\ a \end{pmatrix}$$

(ii) Find a vector equation for the line $2x + 5y - 1 = 0$

$$\underset{\sim}{r} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

any point that lies on the line

looks like $\frac{1}{m}$

(iii) Find the point of intersection of $2x + y + 1 = 0$ and $3x + 5y - 9 = 0$

$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

$$\Rightarrow \quad \lambda = 3 + 5\mu$$
$$-1 - 2\lambda = -3\mu$$

$$-1 - 2(3 + 5\mu) = -3\mu$$
$$7\mu = -7$$
$$\mu = -1$$

$$\therefore \underset{\sim}{r} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

point of intersection is $(-2,3)$

(iv) Find a vector equation for the line joining the points (–1,1) and (4,11). Use this to write parametric equations for any point on the line . Hence find the coordinates of the points where the line meets the parabola $y = x^2$

$$\underset{\sim}{r} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$x = -1 + \lambda$$
$$y = 1 + 2\lambda$$

$$y = x^2$$
$$1 + 2\lambda = (-1 + \lambda)^2$$
$$1 + 2\lambda = 1 - 2\lambda + \lambda^2$$
$$\lambda^2 - 4\lambda = 0$$
$$\lambda(\lambda - 4) = 0$$
$$\lambda = 0 \text{ or } \lambda = 4$$

$\therefore$ parabola meets the line at (–1,1) and (3,9)

(iv) 2022 Extension 2 HSC Q11e)

Let l_1 be the line with equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$

The line l_2 passes through the point $A(-6,5)$ and is parallel to l_1

Find the equation of the line l_2 in the form $y = mx + c$

$$\text{direction vector} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \Rightarrow m = \frac{2}{3}$$

$$y - 5 = \frac{2}{3}(x+6)$$

$$= \frac{2}{3}x + 4$$

$$y = \frac{2}{3}x + 9$$

Vector Equation of a Line in 3D

The equation of the line that passes through A and B is given by;

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$$

where: $\underset{\sim}{a}$ is any point on AB

$\underset{\sim}{b} = \overrightarrow{AB}$ (direction vector)

e.g. (i) find a vector equation of the line through (1,2,3) and (–2,7,4)

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix}$$

(ii) What is its corresponding Cartesian equation?

$$\frac{x-1}{3} = \frac{2-y}{5} = 3 - z$$

(*iii*) 2020 Extension 2 HSC Question 13 b)

Consider the two lines in three dimensions given by

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ and } \underset{\sim}{r} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

By equating components, find the point of intersection of the two lines.

$$3 + \lambda_1 = 3 - 2\lambda_2 \Rightarrow \lambda_1 = -2\lambda_2$$

$$-1 + 2\lambda_1 = -6 + \lambda_2$$

$$-1 - 4\lambda_2 = -6 + \lambda_2$$

$$5\lambda_2 = 5$$

$$\lambda_2 = 1$$

$$7 + \lambda_1 = 2 + 3\lambda_2$$

$$7 - 2\lambda_2 = 2 + 3\lambda_2$$

$$5\lambda_2 = 5$$

$$\lambda_2 = 1$$

substituting into the third component confirms that these lines intersect

as we were told that these lines intersect, there is no need to use the third component

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -5 \\ 5 \end{pmatrix}$$

(iv) Find the vector equation of the line that passes through $(-2,1,4)$ and is parallel to $2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

$$\underset{\sim}{r} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

two lines are parallel if their direction vectors are scalar multiples

(v) Find the vector equation of a line that passes through $(0,2,3)$ and is perpendicular to $\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$

$$\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$a - b + 4c = 0$$

$$\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mu \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

two lines are perpendicular if the dot product of their direction vectors equal zero

$$\lambda a = \mu$$
$$2 + \lambda b = -\mu$$
$$3 + \lambda c = 4\mu$$

$$2 + \lambda b = -\lambda a \Rightarrow \lambda = -\frac{2}{a+b}$$

$$3 + \lambda c = 4\lambda a \Rightarrow \lambda = \frac{3}{4a-c}$$

$$\frac{3}{4a-c} = -\frac{2}{a+b}$$
$$3a + 3b = -8a + 2c$$
$$11a + 3b - 2c = 0$$
$$11a - 11b + 44c = 0$$
$$14b - 46c = 0$$
$$b = \frac{23c}{7}$$

let $c = 7$

$\therefore b = 23$ and $a = -5$

$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 23 \\ 7 \end{pmatrix}$$

if two lines have direction vectors $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ they are;

parallel if $\underset{\sim}{b}_1 = \mu \underset{\sim}{b}_2$, $\mu \in \mathbb{R}$

perpendicular if $\underset{\sim}{b}_1 \cdot \underset{\sim}{b}_2 = 0$

skew if the lines are not parallel and do not intersect

(*vi*) Show that $\underset{\sim}{r}_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\underset{\sim}{r}_2 = \mu \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$ are skewed lines

$1 + \lambda = \mu \ldots.$ ①

$3 + \lambda = 4\mu \ldots$ ②

$-1 = 5\mu \ldots$ ③

equate ① and ③

$\mu = -\frac{1}{5}$ $\quad 1 + \lambda = -\frac{1}{5}$

$\lambda = -\frac{6}{5}$

equate ② and ③

$3 + \lambda = -\frac{4}{5}$

$\lambda = -\frac{19}{5} \neq -\frac{6}{5}$

need to show that there is no point of intersection

thus the lines do not intersect and are not parallel

∴ the lines are skewed

Exercise 5F; 1, 2, 3a, 4b, 5ac, 7a, 9a, 11a, 12b, 14, 15b, 17, 18, 20, 21, 24, 25, 26