Rates of Change & Integration

Integration allows the original quantity to be found from the rate of change.

e.g. (*i*) 2017 Mathematics HSC Q13 d)

The rate at which water flows into a tank is given by

$$\frac{dV}{dt} = \frac{2t}{1+t^2}$$

where V is the volume of water in the tank in litres and t is the time in seconds. Initially the tank is empty.

Find the exact amount of water in the tank after 10 seconds.

$$\frac{dV}{dt} = \frac{2t}{1+t^2}$$

$$\int_0^V dV = \int_0^{10} \frac{2t}{1+t^2} dt$$

Solving as a definite integral

1. separate the variables
2. match the limits of the variables

eliminates the need to find the constant of integration

$$\int_0^V dV = \int_0^{10} \frac{2t}{1+t^2} dt$$

$$\Big[V \Big]_0^V = \Big[\ln(1+t^2) \Big]_0^{10}$$

$$V = \ln\left(\frac{101}{1}\right)$$

$$= \ln(101)$$

$\therefore$ there is ln(101) litres of water left in the tank (*ii*) 2010 Mathematics HSC Q7 a)(ii)

The acceleration of a particle is given by;

$$\ddot{x} = 4\cos 2t$$

where x is displacement in metres and t is time in seconds.

Initially the particle is at the origin with a velocity of 1 ms^{-1}

Find when the particle first comes to rest.

$$\frac{dv}{dt} = 4\cos 2t$$

$$\int_1^0 dv = 4\int_0^t \cos 2t dt$$

$$\left[v\right]_1^0 = 2\left[\sin 2t\right]_0^t$$

$$-1 = 2\sin 2t$$

$$\sin 2t = -\frac{1}{2}$$

$$2t = \frac{7\pi}{6}$$

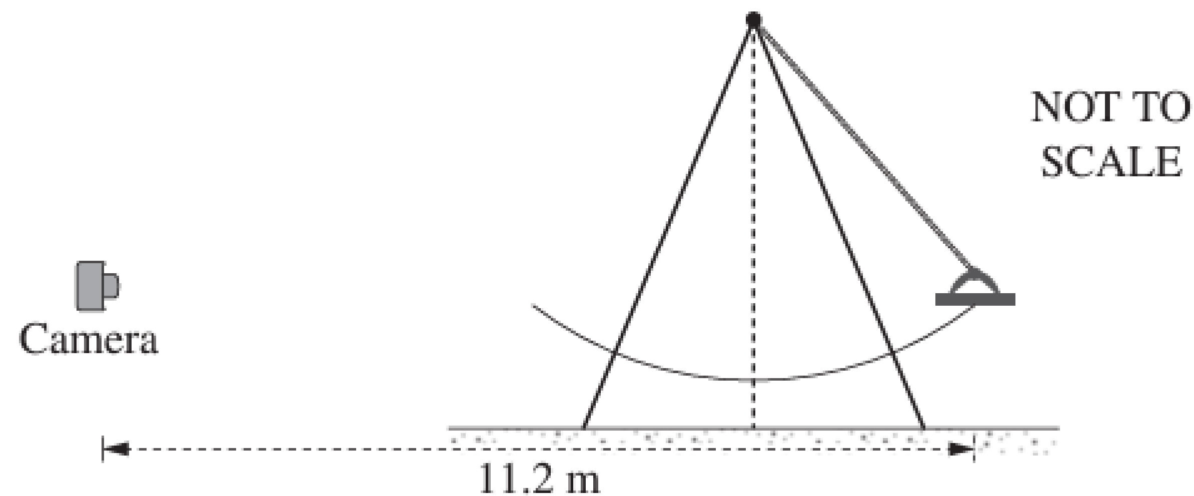
$$t = \frac{7\pi}{12}$$

$\therefore$ the particle first comes to rest after $\frac{7\pi}{12}$ seconds (*iii*) 2023 Mathematics Advanced HSC Q26 a)

A camera films the motion of a swing in a park.

Let $x(t)$ be the horizontal distance, in metres, from the camera to seat of the swing at t seconds.

The seat is released from rest at a horizontal distance of 11.2 m from the camera.



The rate of change of x can be modelled by the equation

$$\frac{dx}{dt} = -1.5\pi \sin\left(\frac{5\pi}{4}t\right)$$

Find an expression for $x(t)$.

$$\frac{dx}{dt} = -1.5\pi \sin\left(\frac{5\pi}{4}t\right)$$

$$\int_{11.2}^{x} dx = -1.5\pi \int_{0}^{t} \sin\left(\frac{5\pi}{4}t\right) dt$$

$$\left[x\right]_{11.2}^{x} = \frac{3\pi}{2} \times \frac{4}{5\pi}\left[\cos\left(\frac{5\pi}{4}t\right)\right]_{0}^{t}$$

$$x - 11.2 = 1.2\cos\left(\frac{5\pi}{4}t\right) - 1.2$$

$$x = 1.2\cos\left(\frac{5\pi}{4}t\right) + 10$$

(iv) 2013 Extension 1 HSC Q13 a)

A spherical raindrop of radius r metres loses water through evaporation at a rate that depends upon its surface area. The rate of change of the volume V of the raindrop is given by

$$\frac{dV}{dt} = -10^{-4} A$$

where t is in seconds and A is the surface area of the rain drop.

a) Show that $\frac{dr}{dt}$ is a constant.

$$\frac{dr}{dt} = ? \quad \frac{dV}{dt} = -10^{-4} A$$

$$\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{dr}{dV}$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\therefore \frac{dV}{dr} = A$$

$$\frac{dr}{dt} = -10^{-4} A \cdot \frac{1}{A}$$

$$= -10^{-4}$$

$\therefore$ radius decreases at a constant rate of 10^{-4} m / s

b) How long does it take for a raindrop of volume 10^{-6} m^3 to completely evaporate?

$$V = \frac{4}{3}\pi r^3$$

$$10^{-6} = \frac{4}{3}\pi r^3$$

$$r^3 = \frac{3\times10^{-6}}{4\pi}$$

$$r = \sqrt[3]{\frac{3\times10^{-6}}{4\pi}}$$

$$\frac{dr}{dt} = -10^{-4}$$

$$-10^4 \int_{\sqrt[3]{\frac{3\times10^{-6}}{4\pi}}}^{0} dr = \int_0^t dt$$

$$t = 10^4\left[r\right]_0^{\sqrt[3]{\frac{3\times10^{-6}}{4\pi}}}$$

$$t = 10^4\sqrt[3]{\frac{3\times10^{-6}}{4\pi}}$$

$$= 62.03504909\ldots$$

$$= 62 \text{ seconds}$$

$\therefore$ it takes approximately 62 seconds to evaporate

Exercise 9F; 2, 3, 6, 8, 9, 11, 12, 13