Vector Equation of a Circle

2D

$$\left| \underset{\sim}{v} - \underset{\sim}{v}_0 \right| = r$$

Is the **vector equation** of a circle in 2D with;

centre: $\underset{\sim}{v}_0$

radius$= r$ units

$$\left| (x - x_0)\underset{\sim}{i} + (y - y_0)\underset{\sim}{j} \right| = r$$

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

cartesian equation of a circle

$$\cos^2\theta + \sin^2\theta = 1$$

$$r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

$$\text{let } (x - x_0) = r\cos\theta \Rightarrow x = x_0 + r\cos\theta$$

$$(y - y_0) = r\sin\theta \Rightarrow y = y_0 + r\sin\theta$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$ **parametric equation of a circle**

e.g. Show that $\left(\underset{\sim}{r} - 2\underset{\sim}{i} + 3\underset{\sim}{j}\right) \cdot \left(\underset{\sim}{r} - 2\underset{\sim}{i} + 3\underset{\sim}{j}\right) = 12$ represents a circle and find its centre and radius

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = |\underset{\sim}{u}|^2$$

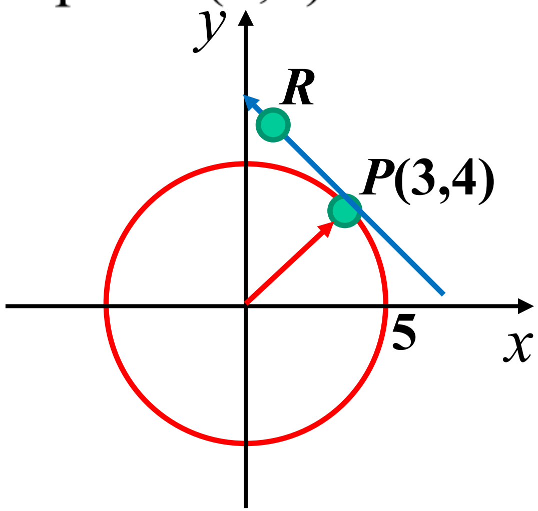
$$\left(\underset{\sim}{r} - 2\underset{\sim}{i} + 3\underset{\sim}{j}\right) \cdot \left(\underset{\sim}{r} - 2\underset{\sim}{i} + 3\underset{\sim}{j}\right) = 12$$

$$\left|\underset{\sim}{r} - 2\underset{\sim}{i} + 3\underset{\sim}{j}\right|^2 = 12$$

$$\left|\underset{\sim}{r} - 2\underset{\sim}{i} + 3\underset{\sim}{j}\right| = 2\sqrt{3}$$

which represents a circle, centre $(2,-3)$ and radius $2\sqrt{3}$ units

(ii) a) Find a vector equation of the tangent to $x^2 + y^2 = 25$ at the point (3,4)



$$\overrightarrow{PR} = \left(\underset{\sim}{r} - 3\underset{\sim}{i} - 4\underset{\sim}{j}\right)$$

$$\overrightarrow{OP} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$$

$$\overrightarrow{PR} \cdot \overrightarrow{OP} = 0 \qquad \text{(radius} \perp \text{tangent)}$$

$$\underline{\left(\underset{\sim}{r} - 3\underset{\sim}{i} - 4\underset{\sim}{j}\right) \cdot \left(3\underset{\sim}{i} + 4\underset{\sim}{j}\right) = 0}$$

b) Find the Cartesian equation of the tangent

$$\left((x-3)\underset{\sim}{i} + (y-4)\underset{\sim}{j}\right) \cdot \left(3\underset{\sim}{i} + 4\underset{\sim}{j}\right) = 0$$

$$3(x-3) + 4(y-4) = 0$$

$$3x - 9 + 4y - 16 = 0$$

$$\underline{3x + 4y - 25 = 0}$$

Vector Equation of a Sphere

3D

$$\left| \underset{\sim}{v} - \underset{\sim}{v}_0 \right| = r$$

Is the **vector equation** of a sphere in 3D with;

centre: $\underset{\sim}{v}_0$

radius= r units

$$\left| (x - x_0)\underset{\sim}{i} + (y - y_0)\underset{\sim}{j} + (z - z_0)\underset{\sim}{k} \right| = r$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

cartesian equation of a sphere

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi \\ r\sin\theta \end{pmatrix}$$

parametric equation of a sphere

e.g. The spheres with equations $(x+2)^2+(y+3)^2+(z-4)^2=16$ and $(x+2)^2+(y+3)^2+(z+2)^2=25$ intersect at a circle.

a) Upon which plane does the circle lie?

$$
\begin{aligned}
(x+2)^2+(y+3)^2+(z-4)^2&=16\\
(x+2)^2+(y+3)^2+(z+2)^2&=25
\end{aligned}
$$

$$
\begin{aligned}
(z+2)^2-(z-4)^2&=9\\
z^2+4z+4-z^2+8z-16&=9\\
12z&=21\\
z&=\frac{7}{4}
\end{aligned}
$$

the two spheres intersect on the plane $z=\frac{7}{4}$

b) Find the centre and radius of the intersecting circle

$$(x + 2)^2 + (y + 3)^2 + \left(\frac{7}{4} - 4\right)^2 = 16$$

$$(x + 2)^2 + (y + 3)^2 + \frac{81}{16} = 16$$

$$(x + 2)^2 + (y + 3)^2 = \frac{175}{16}$$

circle has centre $\left(-2,-3,\frac{7}{4}\right)$ and radius $= \frac{5\sqrt{7}}{4}$ units

(ii) Find the intersection points of the sphere $\left| \underset{\sim}{r} - \underset{\sim}{i} - 4\underset{\sim}{j} \right| = 4$ and the line $\underset{\sim}{r} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k} + \lambda(\underset{\sim}{i} - 2\underset{\sim}{k})$

$$\left| \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k} + \lambda(\underset{\sim}{i} - 2\underset{\sim}{k}) - \underset{\sim}{i} - 4\underset{\sim}{j} \right| = 4$$

$$\left| \lambda\underset{\sim}{i} - 2\underset{\sim}{j} + (3 - 2\lambda)\underset{\sim}{k} \right| = 4$$

$$\lambda^2 + 4 + 9 - 12\lambda + 4\lambda^2 = 16$$

$$5\lambda^2 - 12\lambda - 3 = 0$$

$$\lambda = \frac{12 \pm \sqrt{204}}{10}$$

$$= \frac{6 \pm \sqrt{51}}{5}$$

$$x = 1 + \frac{6 \pm \sqrt{51}}{5}$$

$$y = 2$$

$$z = 3 - \frac{12 \pm 2\sqrt{51}}{5}$$

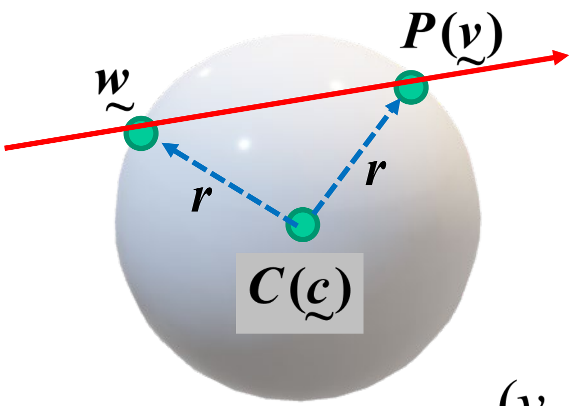
pts of intersection are $\left(\frac{11 + \sqrt{51}}{5}, 2, \frac{3 - 2\sqrt{51}}{5} \right)$ and $\left(\frac{11 - \sqrt{51}}{5}, 2, \frac{3 + 2\sqrt{51}}{5} \right)$

(iii) Let $\underset{\sim}{v}$ be the position vector of a point P on a sphere S with centre C and radius r, so that $|\underset{\sim}{v} - \underset{\sim}{c}| = r$, where $\underset{\sim}{c} = \overrightarrow{OC}$ Do NOT prove this)

a) The equation of the line l through P in the direction of the vector $\underset{\sim}{m}$

$\underset{\sim}{w} = \underset{\sim}{v} + \lambda \underset{\sim}{m}$

Find the values of λ that correspond to the intersection of the line l and the sphere S. Give your answer in terms of $\underset{\sim}{v}$, $\underset{\sim}{c}$ and $\underset{\sim}{m}$



$$|\underset{\sim}{w} - \underset{\sim}{c}| = r$$

$$|\underset{\sim}{v} + \lambda \underset{\sim}{m} - \underset{\sim}{c}| = r$$

$$[(\underset{\sim}{v} - \underset{\sim}{c}) + \lambda \underset{\sim}{m}] \cdot [(\underset{\sim}{v} - \underset{\sim}{c}) + \lambda \underset{\sim}{m}] = r^2$$

$$(\underset{\sim}{v} - \underset{\sim}{c}) \cdot (\underset{\sim}{v} - \underset{\sim}{c}) + 2\lambda(\underset{\sim}{v} - \underset{\sim}{c}) \cdot \underset{\sim}{m} + \lambda^2 \underset{\sim}{m} \cdot \underset{\sim}{m} = r^2$$

$$|\underset{\sim}{v} - \underset{\sim}{c}|^2 + 2\lambda(\underset{\sim}{v} - \underset{\sim}{c}) \cdot \underset{\sim}{m} + \lambda^2 |\underset{\sim}{m}|^2 = r^2$$

$$r^2 + 2\lambda(\underset{\sim}{v} - \underset{\sim}{c}) \cdot \underset{\sim}{m} + \lambda^2 |\underset{\sim}{m}|^2 = r^2$$

$$2\lambda(\underset{\sim}{v} - \underset{\sim}{c})\cdot\underset{\sim}{m} + \lambda^2|\underset{\sim}{m}|^2 = 0$$

$$\lambda[2(\underset{\sim}{v} - \underset{\sim}{c})\cdot\underset{\sim}{m} + \lambda|\underset{\sim}{m}|^2] = 0$$

$$\lambda = 0 \quad \text{or} \quad \lambda = \frac{2\underset{\sim}{m}\cdot(\underset{\sim}{c} - \underset{\sim}{v})}{|\underset{\sim}{m}|^2}$$

b) Deduce that the line l is tangent to the sphere S if and only if $\underset{\sim}{m}\cdot(\underset{\sim}{v} - \underset{\sim}{c}) = 0$. Interpret this result geometrically

If l is a tangent, then there is only one point of intersection

$$\text{i.e. } \underset{\sim}{w} = \underset{\sim}{v}$$

$$\underset{\sim}{v} + \lambda\underset{\sim}{m} = \underset{\sim}{v}$$

$$\lambda = 0$$

$$\frac{-2\underset{\sim}{m}\cdot(\underset{\sim}{c} - \underset{\sim}{v})}{|\underset{\sim}{m}|^2} = 0$$

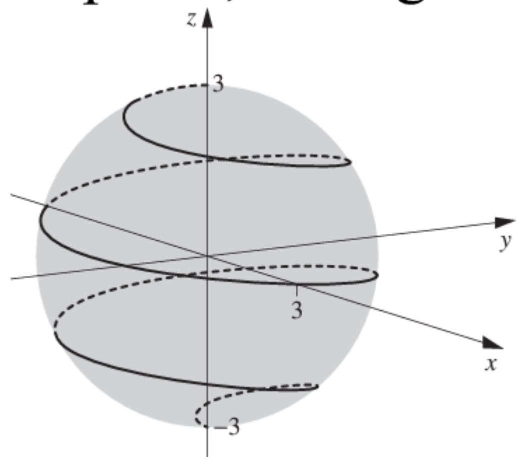
$$\underset{\sim}{m}\cdot(\underset{\sim}{v} - \underset{\sim}{c}) = 0$$

If the dot product equals zero then the tangent must be perpendicular to the radius.

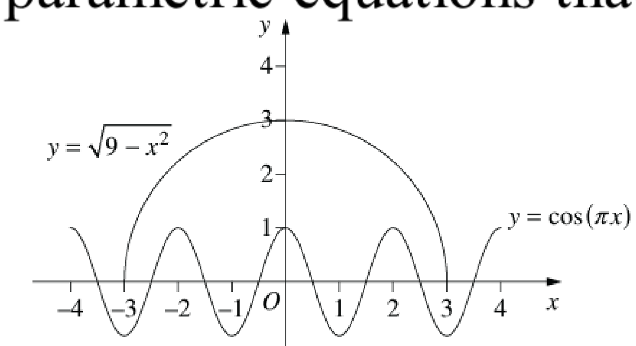
(iii) 2023 Extension 2 HSC 15c)

A curve C spirals 3 times around the sphere centred at the origin and with radius 3, as shown.

A particle is initially at the point (0, 0, –3) and moves along the curve C on the surface of the sphere, ending at the point (0, 0, 3).



By using the diagram below, which shows the graphs of the functions $f(x) = \cos(\pi x)$ and $g(x) = \sqrt{9 - x^2}$, and considering the graph $y = f(x)g(x)$, give a possible set of parametric equations that describe the curve C.



curve is moving from $z = -3$ to $z = 3$, so let $z = t$

consider $f(t)g(t) = \cos(\pi t)\sqrt{9 - t^2}$

$$f(0)g(0) = \cos(0)\sqrt{9}$$
$$= 3$$

when $z = 0$, $x = 3$, so let $x = \cos(\pi t)\sqrt{9 - t^2}$

the equation of the sphere is $x^2 + y^2 + z^2 = 9$

$$\cos^2(\pi t)(9 - t^2) + y^2 + t^2 = 9$$
$$y^2 = 9 - t^2 - \cos^2(\pi t)(9 - t^2)$$
$$= (9 - t^2)\left(1 - \cos^2(\pi t)\right)$$
$$= (9 - t^2)\sin^2(\pi t)$$
$$y = \pm\sin(\pi t)\sqrt{9 - t^2}$$

from the diagram, when initially leaving (0, 0, –3), $y < 0$

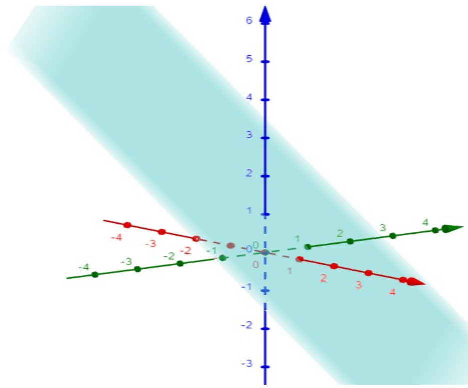
$$\therefore y = -\sin(\pi t)\sqrt{9 - t^2}$$

thus a possible set of parametric equations for the curve C is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos(\pi t)\sqrt{9 - t^2} \\ -\sin(\pi t)\sqrt{9 - t^2} \\ t \end{pmatrix}$$

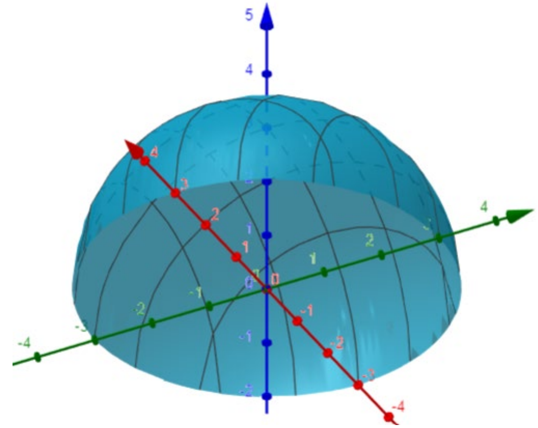
Other Common Graphs in 3D

$$ax + by + cz = d$$



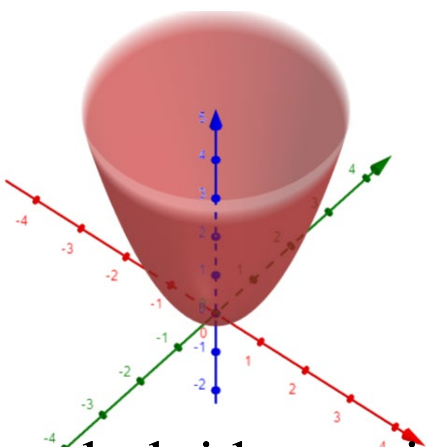
plane

$$z = \sqrt{r^2 - x^2 - y^2}$$



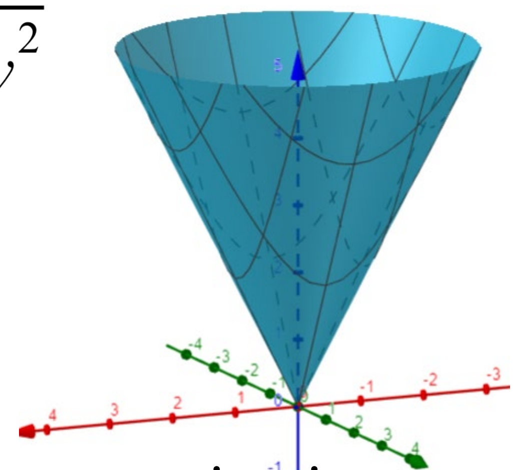
hemisphere, with base on *xy* plane

$$z = x^2 + y^2$$



paraboloid, opening in the *z* direction

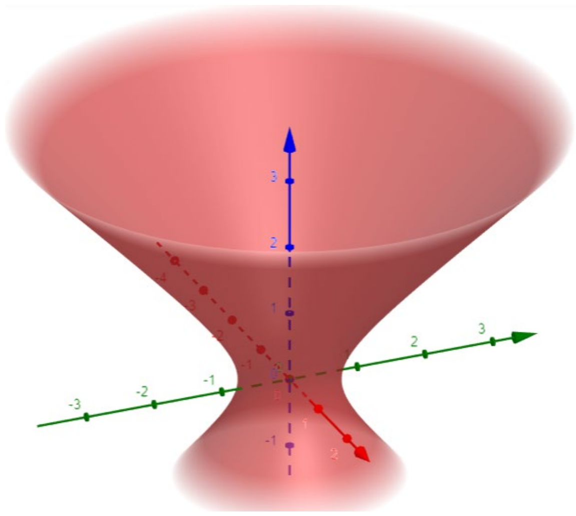
$$z = a\sqrt{x^2 + y^2}$$



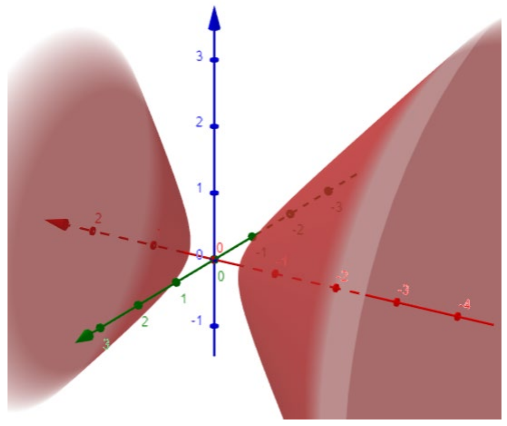
cone, opening in the *z* direction

$$x^2 + y^2 - z^2 = d$$

$$x^2 - y^2 - z^2 = d$$

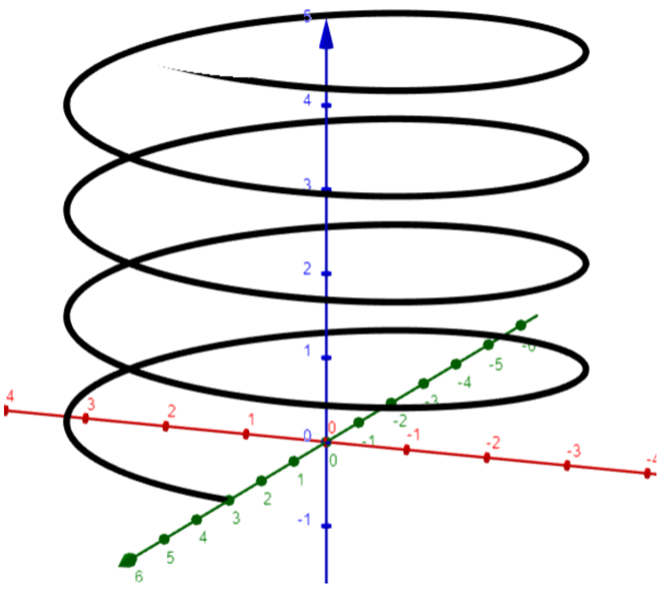


hyperboloid of one sheet



hyperboloid of two sheets

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3\sin 5t \\ 3\cos 5t \\ t \end{pmatrix}$$



helix (spiral)

Exercise 5G; 1, 2, 4, 5b, 7, 8, 9, 10, 11, 13, 14, 15, 17a, 18