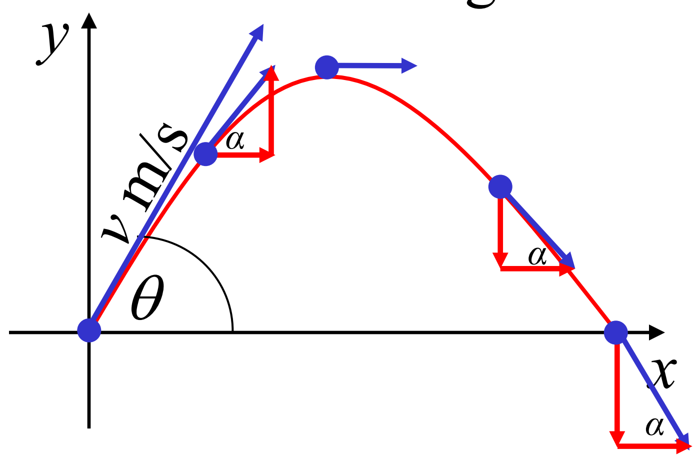
Projectile Motion

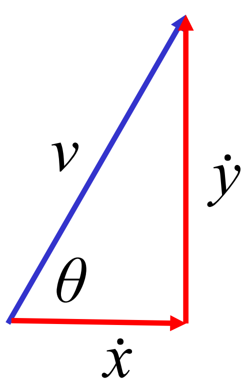
In Extension 1 we consider the two-dimensional motion of a particle projected into the air, subject to gravity only (i.e. we assume that air resistance is negligible), $\ddot{x} = 0 \qquad \ddot{y} = -g\underset{\sim}{j}$

<u>Initial conditions</u> when $t = 0$; particle is projected with a velocity, v ms^{-1}, at an angle of θ to the horizontal



Note:
maximum range

$\theta = 45^\circ$



$$\frac{\dot{x}}{v} = \cos\theta \qquad \frac{\dot{y}}{v} = \sin\theta$$

$$\dot{x} = v\cos\theta \qquad \dot{y} = v\sin\theta$$

$$x = 0 \qquad y = 0$$

$$\frac{d\dot{x}}{dt} = 0 \qquad \frac{d\dot{y}}{dt} = -10$$

$$\int_{v\cos\theta}^{\dot{x}} d\dot{x} = 0 \qquad \int_{v\sin\theta}^{\dot{y}} d\dot{y} = -10\int_0^t dt$$

$$\left[\dot{x}\right]_{v\cos\theta}^{\dot{x}} = 0 \qquad \left[\dot{y}\right]_{v\sin\theta}^{\dot{y}} = -10t$$

$$\dot{x} - v\cos\theta = 0 \qquad \dot{y} - v\sin\theta = -10t$$

$$\dot{x} = v\cos\theta \qquad \dot{y} = v\sin\theta - 10t$$

$$\underset{\sim}{v} = |\underset{\sim}{v}|\cos\theta\,\underset{\sim}{i} + (|\underset{\sim}{v}|\sin\theta - 10t)\underset{\sim}{j}$$

$$\int_0^x dx = \int_0^t v\cos\theta\, dt \qquad \int_0^y dy = \int_0^t (v\sin\theta - 10t)dt$$

$$x = \left[vt\cos\theta\right]_0^t \qquad y = \left[vt\sin\theta - 5t^2\right]_0^t$$

$$x = vt\cos\theta \qquad y = vt\sin\theta - 5t^2$$

$$\underset{\sim}{r} = |\underset{\sim}{v}|t\cos\theta\,\underset{\sim}{i} + \left(|\underset{\sim}{v}|t\sin\theta - 5t^2\right)\underset{\sim}{j}$$

Common Questions

(1) When does the particle hit the ground?

Particle hits the ground when $y = 0$

(2) What is the range of the particle?

(i) find when $y = 0$

(ii) substitute into x

(3) What is the greatest height of the particle?

(i) find when $\dot{y} = 0$

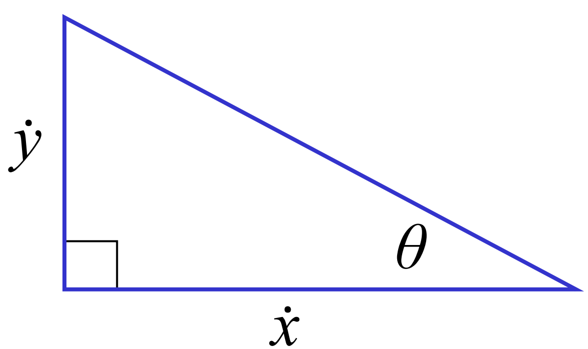
(ii) substitute into y

(4) What angle does the particle make with the ground?

(i) find when $y = 0$

(ii) substitute into $\dot{y}$

(iii) $\tan\theta = \dfrac{\dot{y}}{\dot{x}}$



Summary

A particle undergoing projectile motion obeys

$$\boxed{\ddot{x} = 0 \qquad \text{and} \qquad \ddot{y} = -g}$$

with initial conditions

$$\boxed{\dot{x} = v\cos\theta \qquad \text{and} \qquad \dot{y} = v\sin\theta}$$

e.g. A ball is thrown with an initial velocity of 25 m/s at an angle of $\theta = \tan^{-1}\frac{3}{4}$ to the ground. Determine;.

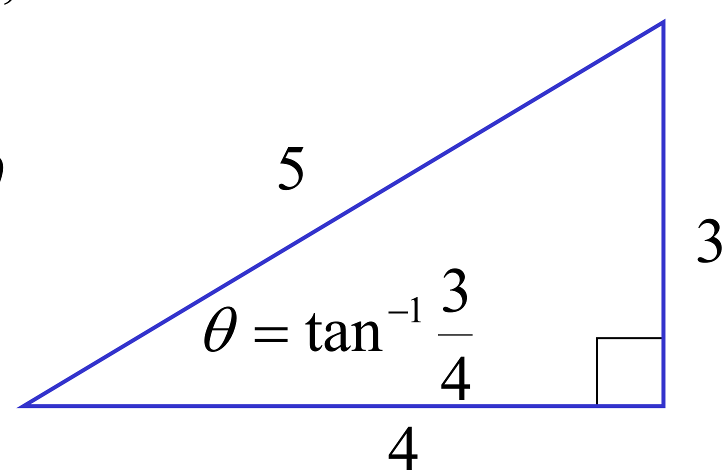
a) greatest height obtained

<u>Initial conditions</u>

$$\dot{x} = v\cos\theta \qquad \dot{y} = v\sin\theta$$

$$\dot{x} = 25\left(\frac{4}{5}\right) \qquad \dot{y} = 25\left(\frac{3}{5}\right)$$

$$= 20\text{m/s} \qquad = 15\text{m/s}$$



$$\ddot{x} = 0$$

$$\frac{d\dot{x}}{dt} = 0$$

$$\int_{20}^{\dot{x}} d\dot{x} = 0$$

$$\left[\dot{x}\right]_{20}^{\dot{x}} = 0$$

$$\dot{x} - 20 = 0$$

$$\dot{x} = 20$$

$$\frac{dx}{dt} = 20$$

$$\int_{0}^{x} dx = \int_{0}^{t} 20dt$$

$$\left[x\right]_{0}^{x} = \left[20t\right]_{0}^{t}$$

$$x = 20t$$

$$\ddot{y} = -10$$

$$\frac{d\dot{y}}{dt} = -10$$

$$\int_{15}^{\dot{y}} d\dot{y} = -\int_{0}^{t} 10dt$$

$$\left[\dot{y}\right]_{15}^{\dot{y}} = \left[10t\right]_{t}^{0}$$

$$\dot{y} - 15 = 0 - 10t$$

$$\dot{y} = 15 - 10t$$

$$\frac{dy}{dt} = 15 - 10t$$

$$\int_{0}^{y} dy = \int_{0}^{t} 15 - 10tdt$$

$$\left[y\right]_{0}^{y} = \left[15t - 5t^2\right]_{0}^{t}$$

$$y = 15t - 5t^2$$

greatest height occurs when $\dot{y} = 0$

$$-10t + 15 = 0$$

$$t = \frac{3}{2}$$

when $t = \frac{3}{2}$, $y = -5\left(\frac{3}{2}\right)^2 + 15\left(\frac{3}{2}\right)$

$$= \frac{45}{4}$$

$\therefore$ greatest height is $11\frac{1}{4}$ m above the ground

b) range

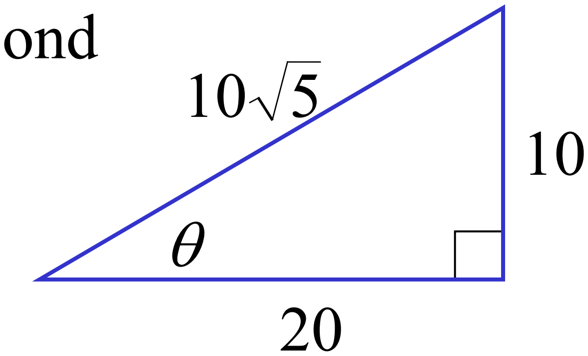
time of flight is 3 seconds

when $t = 3$, $x = 20(3)$

$$= 60$$

$\therefore$ range is 60m

c) velocity and direction of the ball after $\frac{1}{2}$ second



when $t=\frac{1}{2}, \dot{x}=20$

$$\dot{y}=-10\left(\frac{1}{2}\right)+15$$
$$=10$$

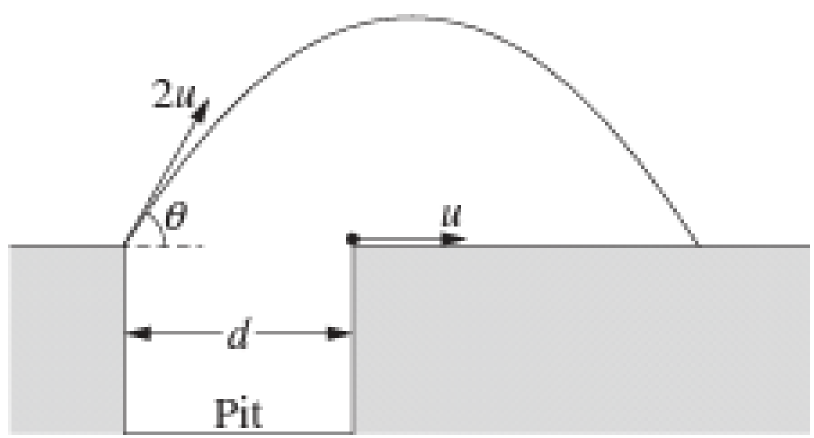
$$\tan\theta=\frac{1}{2}$$
$$\theta=26^\circ 34'$$

$\therefore$ after $\frac{1}{2}$ second, velocity $=10\sqrt{5}$m/s and it is traveling at an angle of $26^\circ 34'$ to the horizontal

(ii) **2022 Extension 1 HSC Question 14 b)**

A video game designer wants to include an obstacle in the game they are developing. The player will reach one side of a pit and must shoot a projectile to hit a target on the other side of the pit in order to be able to cross. However, the instant the player shoots, the target begins to move away from the player at a constant speed that is half the initial speed of the projectile shot by the player, as hown in the diagram below.

The initial distance between the player and the target is d, the initial speed of the projectile is $2u$ and it is launched at an angle of θ to the horizontal. The acceleration due to gravity is g. The launch angle is the ONLY parameter that the player can change.



Taking the position of the player when the projectile is launched as the origin, the positions of the projectile and target at time *t* after the projectile is launched are as follows.

$$\underset{r_p}{\rightarrow} = \begin{pmatrix} 2ut\cos\theta \\ 2ut\sin\theta - \frac{g}{2}t^2 \end{pmatrix} \qquad \text{Projectile}$$

$$\underset{r_T}{\rightarrow} = \begin{pmatrix} d + ut \\ 0 \end{pmatrix} \qquad \text{Target}$$

Show that, for the player to have a chance of hitting the target, *d* must be less than 37% of the maximum possible range of the projectile

projectile hits the ground when $\underset{r_p}{\rightarrow} = \begin{pmatrix} 2ut\cos\theta \\ 0 \end{pmatrix}$

i.e. $2ut\sin\theta - \frac{1}{2}gt^2 = 0$

$$t\left(2u\sin\theta - \frac{1}{2}gt\right) = 0$$

$t = 0$ or $t = \dfrac{4u\sin\theta}{g}$ $\quad\therefore$ projectile hits the ground when $t = \dfrac{4u\sin\theta}{g}$

when $t=\dfrac{4u\sin\theta}{g}$, $x_p=2u\times\dfrac{4u\sin\theta}{g}\times\cos\theta$ $\qquad x_T=d+u\times\dfrac{4u\sin\theta}{g}$

$$=\frac{8u^2\sin\theta\cos\theta}{g} \qquad\qquad =d+\frac{4u^2\sin\theta}{g}$$

$$=\frac{4u^2\sin 2\theta}{g} \quad \Rightarrow \text{ maximum range is } \frac{4u^2}{g}$$

In order to hit the target $x_T\leq x_p$

$$\text{i.e. } d+\frac{4u^2\sin\theta}{g}\leq\frac{4u^2\sin 2\theta}{g}$$

$$d\leq\frac{4u^2}{g}(\sin 2\theta-\sin\theta)$$

d will be maximised when $\sin 2\theta-\sin\theta$ is maximised

$$f(\theta)=\sin 2\theta-\sin\theta$$

$$\begin{aligned}f'(\theta)&=2\cos 2\theta-\cos\theta\\&=2(2\cos^2\theta-1)-\cos\theta\\&=4\cos^2\theta-\cos\theta-2\end{aligned}$$

stationary points occur when $f'(\theta) = 0$

i.e. $4\cos^2\theta - \cos\theta - 2 = 0$

$$\cos\theta = \frac{1 \pm \sqrt{33}}{8}$$

$$\cos\theta = \frac{1 + \sqrt{33}}{8} \qquad (\theta \text{ is acute})$$

$$= 0.8431....$$

$$\sin\theta = \sqrt{1 - 0.8431^2}$$
$$= 0.5378...$$

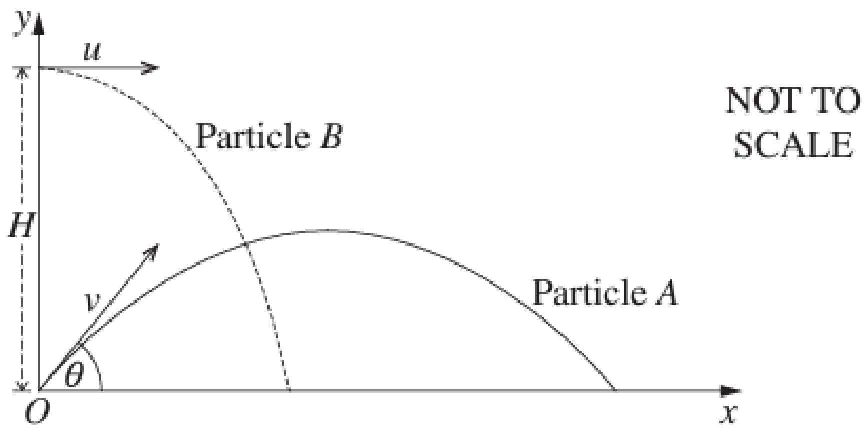
$$d \leq \frac{4u^2}{g}[2(0.5378)(0.8431) - 0.5378)]$$

$$= \frac{4u^2}{g}(0.3690)$$

$\therefore$ d must be 37% of the maximum range.

(iii) **2023 Extension 1 HSC Question 13 b)**

Particle A is projected from the origin with initial speed v ms^{-1} at an angle θ with the horizontal plane. At the same time, particle B is projected horizontally with initial speed u ms^{-1} from a point that is H metres above the origin, as shown in the diagram.



The position vector of particle A, t seconds after it is projected, is given by

$$\mathbf{r}_A(t) = \begin{pmatrix} vt\cos\theta \\ vt\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$$

The position vector of particle B, t seconds after it is projected, is given by

$$\mathbf{r}_B(t) = \begin{pmatrix} ut \\ H - \frac{1}{2}gt^2 \end{pmatrix}$$

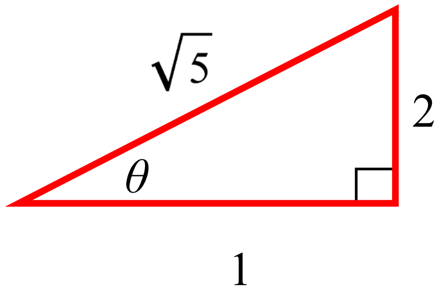
The angle θ is chosen so that $\tan\theta = 2$.

The two particles collide.

a) By first showing that $\cos\theta = \dfrac{1}{\sqrt{5}}$, verify that $v = \sqrt{5}\,u$

$\tan\theta = 2$

$\therefore \cos\theta = \dfrac{1}{\sqrt{5}}$



When the particles collide their horizontal displacements are equal

i.e. $vt\cos\theta = ut$

$$\frac{vt}{\sqrt{5}} = ut$$

$$\underline{v = \sqrt{5}\,u}$$

b) Show that the particles collide at time $T = \dfrac{H}{2u}$

When $t = T$, $y_A = y_B$ i.e. $vT\sin\theta - \frac{1}{2}gT^2 = H - \frac{1}{2}gT^2$

$$\sqrt{5}\,uT \times \frac{2}{\sqrt{5}} = H$$

$$\underline{T = \frac{H}{2u}}$$

When the particles collide, their velocity vectors are perpendicular.

c) Show that $H = \dfrac{2u^2}{g}$ $\qquad \dot{\boldsymbol{r}}_A(t) = \begin{pmatrix} v\cos\theta \\ v\sin\theta - gt \end{pmatrix} \qquad \dot{\boldsymbol{r}}_B(t) = \begin{pmatrix} u \\ -gt \end{pmatrix}$

When $t = T$; the velocity vectors are perpendicular

$$\text{i.e. } \dot{\boldsymbol{r}}_A(T) \cdot \dot{\boldsymbol{r}}_B(T) = 0$$

$$(v\cos\theta)(u) + (v\sin\theta - gT)(-gT) = 0$$

$$uv\cos\theta - gTv\sin\theta + g^2T^2 = 0$$

$$(v\cos\theta)(u) + (v\sin\theta - gT)(-gT) = 0$$

$$uv\cos\theta - gTv\sin\theta + g^2T^2 = 0$$

$$u \times \sqrt{5}\,u \times \frac{1}{\sqrt{5}} - gT \times \sqrt{5}\,u \times \frac{2}{\sqrt{5}} + g^2T^2 = 0$$

$$u^2 - 2ugT + g^2T^2 = 0$$

$$(u - gT)^2 = 0$$

$$T = \frac{u}{g}$$

$$T = \frac{u}{g}$$

$$\frac{H}{2u} = \frac{u}{g} \qquad T = \frac{H}{2u} \text{ from b)}$$

$$H = \frac{2u^2}{g}$$

d) Prior to the collision, the trajectory of particle A was a parabola.

Find the height of the vertex of that parabola above the horizontal plane. Give your answer in terms of H.

Maximum height occurs when $\dot{y}_A = 0$

$$\sqrt{5}\,u \times \frac{2}{\sqrt{5}} - gt = 0$$

$$t = \frac{2u}{g}$$

$$= \frac{H}{u}$$

$$\text{when } t=\frac{H}{u}\,; \qquad y_A=\sqrt{5}\,u\times\frac{H}{u}\times\frac{2}{\sqrt{5}}-\frac{1}{2}g\times\left(\frac{H}{u}\right)^2$$

$$=2H-\frac{gH^2}{2u^2}$$

Exercise 10A; 1, 3, 5, 7, 9, 10, 11, 13, 14, 15, 16, 17