Simple Harmonic Motion

A particle that moves back and forward in such a way that its acceleration at any instant is directly proportional to its distance from a fixed point, is said to undergo **Simple Harmonic Motion (SHM)**

$$\ddot{x} \alpha x$$

$$\ddot{x} = kx$$

$$\ddot{x} = -n^2 x \quad \text{(constant needs to be negative)}$$

If a particle undergoes SHM, then it obeys;

$$\ddot{x} = -n^2 x$$

$$v \frac{dv}{dx} = -n^2 x$$

$$\int_0^v v\,dv = -n^2 \int_a^x x\,dx$$

$$\left[v^2\right]_0^v = -n^2\left[x^2\right]_a^x \qquad (a = \text{amplitude})$$

$$v^2 = n^2(a^2 - x^2)$$

$$v = \pm n\sqrt{a^2 - x^2}$$

NOTE:

$$v^2 \geq 0$$

$$a^2 - x^2 \geq 0$$

$$-a \leq x \leq a$$

$\therefore$ Particle travels back and forward between $x = -a$ and $x = a$

$$\frac{dx}{dt} = -n\sqrt{a^2 - x^2}$$

$$\frac{dt}{dx} = \frac{-1}{n\sqrt{a^2 - x^2}}$$

$$\int_0^t dt = \frac{1}{n}\int_a^x \frac{-1}{\sqrt{a^2 - x^2}} dx$$

$$t = \frac{1}{n}\left[\cos^{-1}\frac{x}{a}\right]_a^x$$

$$= \frac{1}{n}\left\{\cos^{-1}\frac{x}{a} - \cos^{-1} 1\right\}$$

$$= \frac{1}{n}\cos^{-1}\frac{x}{a}$$

$$nt = \cos^{-1}\frac{x}{a}$$

$$\frac{x}{a} = \cos nt$$

$$\underline{x = a\cos nt}$$

If when $t = 0$;

$x = \pm a$, choose - ve and $\cos^{-1}$

$x = 0$, choose + ve and $\sin^{-1}$

In general;

A particle undergoing SHM obeys

$$\boxed{\ddot{x} = -n^2 x}$$

$v^2 = n^2\left(a^2 - x^2\right) \Rightarrow$ allows us to find path of the particle

$$\boxed{x = a\cos nt \qquad \text{OR } x = a\sin nt}$$

where $a =$ amplitude

the particle has;

period : $T = \dfrac{2\pi}{n}$ (time for one oscillation)

frequency : $f = \dfrac{1}{T}$ (number of oscillations per time period) e.g. (i) A particle, P, moves on the x axis according to the law $x = 4\sin 3t$.

a) Show that P is moving in SHM and state the period of motion.

$$x = 4\sin 3t$$

$$\dot{x} = 12\cos 3t$$

$$\ddot{x} = -36\sin 3t$$

$$= -9x$$

$\therefore$ particle moves in SHM

$$T = \frac{2\pi}{3}$$

$\therefore$ period of motion is $\frac{2\pi}{3}$ seconds

b) Find the interval in which the particle moves and determine the greatest speed.

$\therefore$ particle moves along the interval $-4 \leq x \leq 4$

and the greatest speed is 12 units/s

(ii) A particle moves so that its acceleration is given by $\ddot{x} = -4x$

Initially the particle is 3cm to the right of O and traveling with a velocity of 6cm/s.

Find the interval in which the particle moves and determine the greatest acceleration.

$$v\frac{dv}{dx} = -4x$$

$$\int_6^v v\,dv = \int_3^x -4x\,dx$$

$$\left[v^2\right]_6^v = -4\left[x^2\right]_3^x$$

$$v^2 - 36 = -4x^2 + 36$$

$$v^2 = -4x^2 + 72$$

But $v^2 \geq 0$

$$-4x^2 + 72 \geq 0$$

$$x^2 \leq 18$$

$$\underline{-3\sqrt{2} \leq x \leq 3\sqrt{2}}$$

$$\text{when } x = 3\sqrt{2}, \ddot{x} = -4\left(3\sqrt{2}\right)$$

$$= -12\sqrt{2}$$

$\therefore$ <u>greatest acceleration is $12\sqrt{2}$ cm/s^2</u>

(iii) 2012 Extension 1 HSC Q 13c)

A particle is moving in a straight line according to the equation

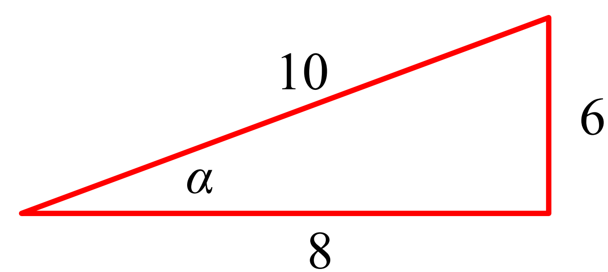
$$x = 5 + 6\cos 2t + 8\sin 2t$$

where x is the displacement in metres and t is the time in seconds

a) Prove that the particle is moving in SHM by showing that

$$\ddot{x} = -n^2(x - c)$$

$$6\cos 2t + 8\sin 2t = 10\sin(2t + \alpha)$$



$$x = 5 + 10\sin(2t + \alpha)$$

$$\alpha = \tan^{-1}\frac{3}{4} = 0.6435\ldots$$

$$\dot{x} = 20\cos(2t + \alpha)$$

$$\begin{aligned}\ddot{x} &= -40\sin(2t + \alpha)\\ &= -4(5 + 10\sin(2t + \alpha) - 5)\\ &= -4(x - 5)\end{aligned}$$

$\therefore$ particle moves in SHM as it is in the form $\ddot{x} = -n^2 x$

b) When is the displacement of the particle zero for the first time?

$$5 + 10\sin(2t + \alpha) = 0$$

$$\sin(2t + \alpha) = -\frac{1}{2}$$

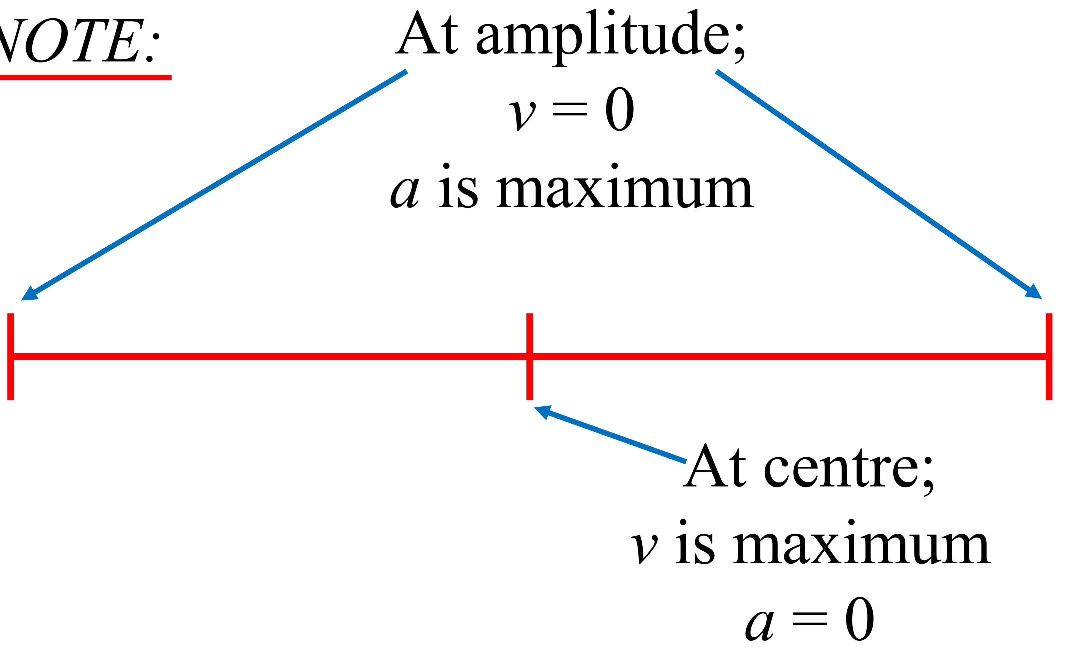
$$2t + \alpha = \frac{7\pi}{6}$$

$$2t = \frac{7\pi}{6} - \alpha$$

$$t = \frac{7\pi}{12} - \frac{\alpha}{2}$$

$$t = 1.5108\ldots$$

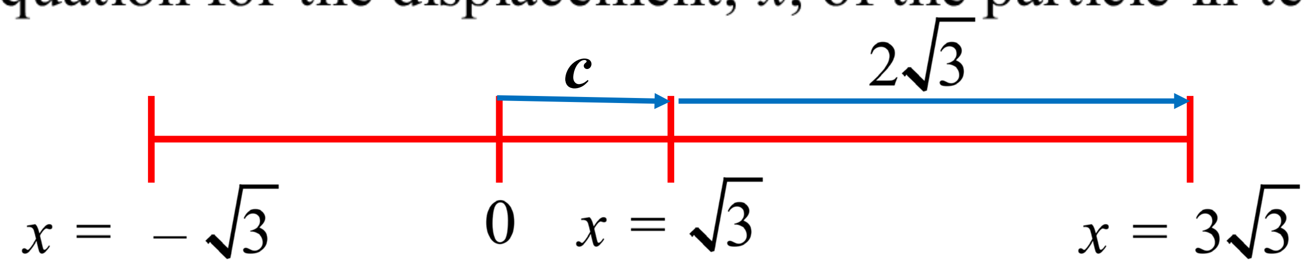
NOTE:



$\therefore$ particle's displacement is first zero after 1.5 seconds (*iv*) 2020 Extension 2 HSC Q 13a)

A particle is undergoing simple harmonic motion with period $\frac{\pi}{3}$.
The central point of motion of the particle is at $x = \sqrt{3}$. When $t = 0$ the particle has its maximum displacement of $2\sqrt{3}$ from the central point of motion.

Find an equation for the displacement, x, of the particle in terms of t.



Particle starts at the amplitude, so the equation is in the form

$$x = a \cos nt + c$$

$$T = \frac{\pi}{3}$$

$$\frac{2\pi}{n} = \frac{\pi}{3}$$

$$n = 6$$

$$c = \sqrt{3}$$

$$a = 2\sqrt{3}$$

$$\therefore x = 2\sqrt{3}\cos 6t + \sqrt{3}$$

(*v*) 2021Extension 2 HSC Q 13d)

An object is moving in simple harmonic motion along the x-axis. The acceleration of the object is given by $\ddot{x} = -4(x - 3)$ where x is its displacement from the origin measured in metres, after t seconds.

Initially, the object is 5.5 metres to the right of the origin and moving towards the origin. The object has a speed of 8 ms^{-1} as it passes through the origin.

a) Between which two values of x is the particle oscillating?

$$\ddot{x} = -4(x - 3)$$

$$v\frac{dv}{dx} = -4(x - 3)$$

$$\int_8^v v\,dv = 4\int_0^x (3 - x)\,dx$$

$$\left[v^2\right]_8^v = 8\left[3x - \frac{1}{2}x^2\right]_0^x$$

$$v^2 - 64 = 8\left(3x - \frac{1}{2}x^2\right)$$

$$v^2 = 64 + 24x - 4x^2$$

$$\text{but } v^2 \geq 0$$

$$\therefore x^2 - 6x - 16 \leq 0$$

$$(x-3)^2 \leq 25$$

$$-5 \leq x - 3 \leq 5$$

$$\underline{-2 \leq x \leq 8}$$

b) Find the first value of t for which $x = 0$, giving the answer correct to 2 decimal places.

from the reference sheet: $\ddot{x} = -n^2(x - c) \Rightarrow n = 2\ ,\ c = 3$

$$x = a\cos(nt + \alpha) + c \Rightarrow \text{from a) } a = 5$$

$$x = 5\cos(2t + \alpha) + 3$$

$$\text{when } t = 0\ ,\ x = \frac{11}{2}\ ;\quad \frac{11}{2} = 5\cos\alpha + 3$$

$$5\cos\alpha = \frac{5}{2}$$

$$\cos\alpha = \frac{1}{2}$$

$$\alpha = \frac{\pi}{3}$$

$$x = 0\ ;\ 5\cos\left(2t + \frac{\pi}{3}\right) + 3 = 0$$

$$\cos\left(2t + \frac{\pi}{3}\right) = -\frac{3}{5}$$

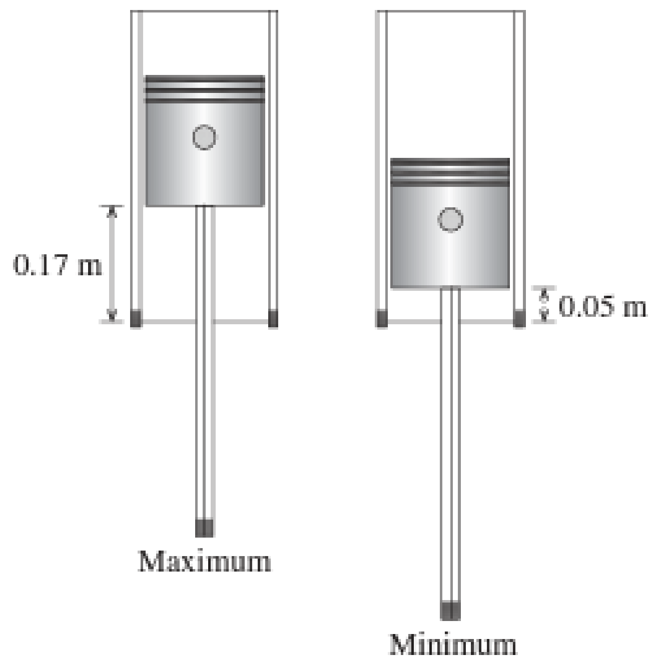
$$2t + \frac{\pi}{3} = 2.214$$

$$2t = 1.167$$

$$\underline{t = 0.58} \quad \text{(to 2 dp)}$$

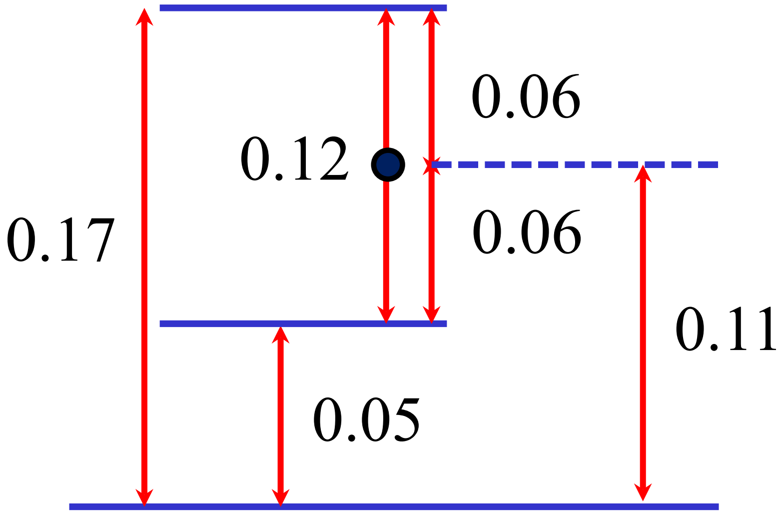
(vi) 2022Extension 2 HSC Q 15b)

The diagram shows two positions of a single piston in the cylinder chamber of a motorcycle. The piston moves vertically, in simple harmonic motion, between a maximum height of 0.17 metres and a minimum height of 0.05 metres



The mass of the piston is 0.8 kg. The piston completes 40 cycles per second.

What is the resultant force on the piston, in newtons, that produces the maximum acceleration of the piston?



centre of motion is at $x = 0.11$

frequency = 40 cycles/second

$$\therefore \text{period} = \frac{1}{40}$$

$$\frac{2\pi}{n} = \frac{1}{40}$$

$$n = 80\pi$$

motion of the piston can be modelled by

$$\ddot{x} = -6\,400\pi^2(x - 0.11)$$

maximum acceleration occurs at the amplitude

$$\text{i.e. when } x = 0.05$$

$$\ddot{x}_{max} = -6\,400\pi^2(0.05 - 0.11)$$

$$= 384\pi^2$$

$$F_{max} = m\ddot{x}_{max}$$

$$= 0.8 \times 384\pi^2$$

$$= 3\,031.942\ldots$$

A force of 3032 N will produce the maximum acceleration of the piston.

Exercise 6B; 1, 5, 7, 8, 10, 12, 13, 17, 18, 21, 23

(start with trig,
prove SHM or are told)

Exercise 6C; 1, 4, 5b, 6b, 8, 9, 10, 11, 14 a, b(*ii,iv*), 16, 18, 19, 23

(start with $\ddot{x} = -n^2x$ *)*