Further Projectile Motion

e.g. A stone is thrown so that it will hit a bird at the top of a pole. However, at the instant the stone is thrown, the bird flies away in a horizontal straight line at a speed of 10 m/s.

The stone reaches a height double that of the pole and, in its descent, touches the bird.

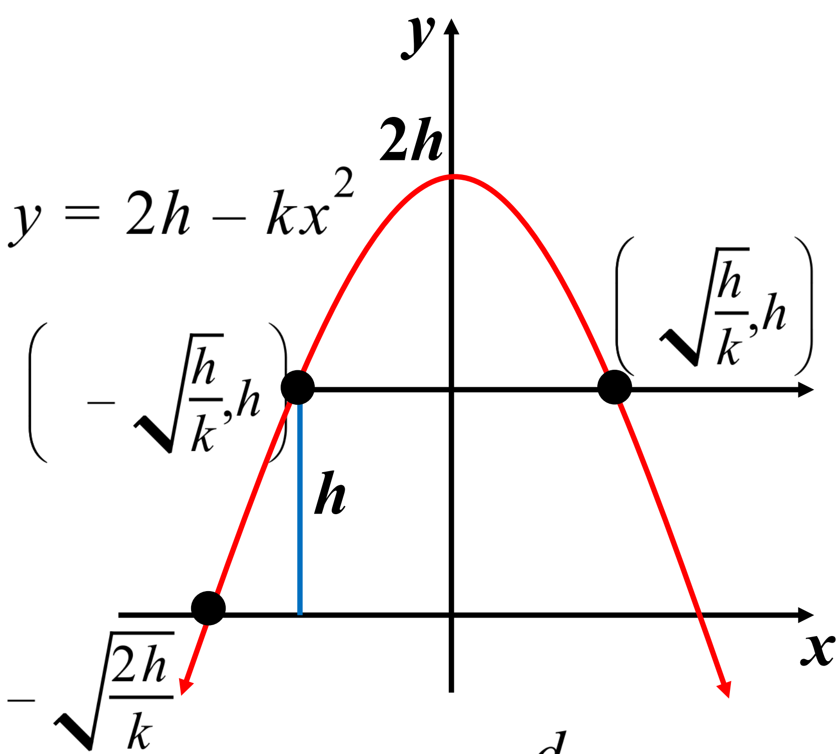
Find the horizontal component of the velocity of the stone.

Assuming there is no air resistance, the path of a projectile will be a parabola

sometimes your knowledge of the quadratic function, can assist in solving projectile motion questions

The greatest height will occur at the vertex of the parabola, so let the coordinates of the vertex be $(0,2h)$.

Thus the equation of the parabola will be $y = 2h - kx^2$



$$\text{time} = \frac{d}{s}$$

$$= \frac{2\sqrt{\frac{h}{k}}}{10}$$

$$= \frac{1}{5}\sqrt{\frac{h}{k}}$$

when $y = h$, $h = 2h - kx^2$

$$kx^2 = h$$

$$x^2 = \frac{h}{k}$$

$$x = \pm\sqrt{\frac{h}{k}}$$

so the bird flies a total of $2\sqrt{\frac{h}{k}}$ metres at 10 ms^{-1}

when $y = 0$, $0 = 2h - kx^2$

$$kx^2 = 2h$$

$$x^2 = \frac{2h}{k}$$

$$x = \pm\sqrt{\frac{2h}{k}}$$

the rock travels $\sqrt{\frac{2h}{k}} + \sqrt{\frac{h}{k}}$

$$= \sqrt{\frac{h}{k}}\,(\sqrt{2} + 1) \text{ metres horizontally}$$

$$s = \frac{d}{t}$$

$$= \frac{\sqrt{\frac{h}{k}}\,(\sqrt{2} + 1)}{\frac{1}{5}\sqrt{\frac{h}{k}}}$$

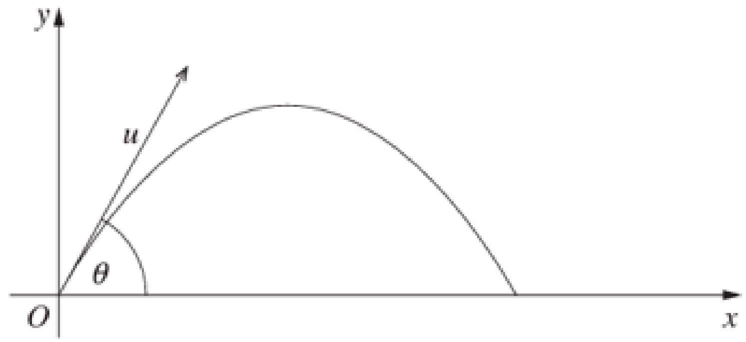
$$= 5(\sqrt{2} + 1)\ ms^{-1}$$

$\therefore$ horizontal velocity of the stone is $5(\sqrt{2} + 1)$ ms^{-1}

Projectile Motion & Resistance

e.g. (i) **2021 Extension 2 HSC Q16 b)**

A particle which is projected from the origin with initial speed u ms^{-1} at an angle of θ to the positive x-axis lands on the x-axis, as shown in the diagram. The particle is subject to an acceleration due to gravity of g ms^{-2}.



The position vector of the particle $\underset{\sim}{r}(t)$, where t is the time in seconds after the particle is projected, is given by

$$\underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ -\dfrac{gt^2}{2} + ut\sin\theta \end{pmatrix}$$ (Do NOT prove this)

For some value(s) of θ there will be two times during the flight when the particle's position vector is perpendicular to its velocity vector.

Find the value(s) of θ for which this occurs, justifying that both times occur during the time of flight.

$$\underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ -\frac{gt^2}{2} + ut\sin\theta \end{pmatrix} \qquad \underset{\sim}{v}(t) = \begin{pmatrix} u\cos\theta \\ -gt + u\sin\theta \end{pmatrix}$$

$$\underset{\sim}{r} \perp \underset{\sim}{v} \Rightarrow \underset{\sim}{r} \cdot \underset{\sim}{v} = 0$$

$$u^2 t\cos^2\theta + \frac{1}{2}g^2t^3 - \frac{1}{2}gut^2\sin\theta - gut^2\sin\theta + u^2t\sin^2\theta = 0$$

$$u^2t + \frac{1}{2}g^2t^3 - \frac{3}{2}gut^2\sin\theta = 0$$

however $t > 0$ as it is during the time of flight

$$g^2t^2 - 3gut\sin\theta + 2u^2 = 0$$

as there are two distinct times this happens, $\Delta > 0$

$$9g^2u^2\sin^2\theta - 8g^2u^2 > 0$$

$$9\sin^2\theta - 8 > 0$$

$$\sin^2\theta > \frac{8}{9}$$

$$\therefore \frac{8}{9} < \sin^2\theta < 1 \qquad (\theta \text{ is acute})$$

$$\sqrt{\frac{8}{9}} < \sin\theta < 1$$

$$\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) < \theta < 90°$$

$$71° < \theta < 90° \quad \text{(to the nearest degree)}$$

particle hits the ground when $y = 0$ i.e. $-\frac{1}{2}gt^2 + ut\sin\theta = 0$

$$-\frac{1}{2}gt + u\sin\theta = 0$$

$$t = \frac{2u\sin\theta}{g}$$

so both values of θ must occur when

$$0 < t \le \frac{2u\sin\theta}{g}$$

$$g^2t^2 - 3gut\sin\theta + 2u^2 = 0$$

$$t = \frac{3gu\sin\theta \pm \sqrt{9g^2u^2\sin^2\theta - 8g^2u^2}}{2g^2}$$

$$= \frac{3u\sin\theta \pm u\sqrt{9\sin^2\theta - 8}}{2g}$$

now $3u\sin\theta - u\sqrt{9\sin^2\theta - 8} > 3u\sin\theta - u\sqrt{9\sin^2\theta}$

$$= 0$$

$\therefore$ both times are > 0

the larger value is

$$t = \frac{3u\sin\theta + u\sqrt{9\sin^2\theta - 8}}{2g}$$

$$< \frac{3u\sin\theta + u\sqrt{9\sin^2\theta - 8\sin^2\theta}}{2g} \qquad (\sin^2\theta \le 1)$$

$$t < \frac{3u\sin\theta + u\sin\theta}{2g}$$
$$= \frac{2u\sin\theta}{g}$$

so both times occur within the time of flight

$$71° < \theta < 90°$$

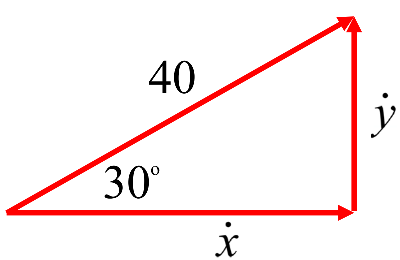
(*ii*) 2023 Extension 2 HSC Q13 c)

A particle of mass 1 kg is projected from the origin with speed 40 ms^{-1} at an angle of 30° to the horizontal plane.

a) Use the information above to show that the initial velocity of the particle is

$$\mathbf{v}(0) = \begin{pmatrix} 20\sqrt{3} \\ 20 \end{pmatrix}$$

$$\frac{\dot{x}}{40} = \cos 30° \qquad \frac{\dot{y}}{40} = \sin 30°$$



$$\dot{x} = 40 \times \frac{\sqrt{3}}{2} \qquad \dot{y} = 40 \times \frac{1}{2}$$
$$= 20\sqrt{3} \qquad = 20$$

$$\therefore \mathbf{v}(0) = \begin{pmatrix} 20\sqrt{3} \\ 20 \end{pmatrix}$$

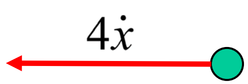
The forces acting on the particle are gravity and air resistance. The air resistance is proportional to the velocity with a constant of proportionality 4. Let the acceleration due to gravity be 10 ms^{-2}.

The position vector of the particle, at time t seconds after the particle is projected, is $\mathbf{r}(t)$ and the velocity vector is $\mathbf{v}(t)$.

b) Show that $\mathbf{v}(t) = \begin{pmatrix} 20\sqrt{3}\,e^{-4t} \\ \dfrac{45}{2}e^{-4t} - \dfrac{5}{2} \end{pmatrix}$

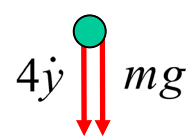
start by drawing force diagrams even for the simplest of questions

horizontal forces



$$m\ddot{x} = -4\dot{x}$$
$$\ddot{x} = -4\dot{x}$$

vertical forces



$$m\ddot{y} = -4\dot{y} - mg$$
$$\ddot{y} = -4\dot{y} - 10$$

$$\frac{d\dot{x}}{dt} = -4\dot{x}$$

$$\int_{20\sqrt{3}}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -4\int_{0}^{t} dt$$

$$\left[\ln\ \dot{x}\ \right]_{20\sqrt{3}}^{\dot{x}} = -4t$$

$$\ln\left(\frac{\dot{x}}{20\sqrt{3}}\right) = -4t$$

$$\frac{\dot{x}}{20\sqrt{3}} = e^{-4t}$$

$$\dot{x} = 20\sqrt{3}\,e^{-4t}$$

$$\frac{d\dot{y}}{dt} = -4\dot{y} - 10$$

$$\int_{20}^{\dot{y}} \frac{d\dot{y}}{4\dot{y}+10} = -\int_{0}^{t} dt$$

$$\frac{1}{4}\left[\ln\left(4\dot{y}+10\right)\right]_{20}^{\dot{y}} = -t$$

$$\ln\left(\frac{4\dot{y}+10}{90}\right) = -4t$$

$$\frac{2\dot{y}+5}{45} = e^{-4t}$$

$$2\dot{y}+5 = 45e^{-4t}$$

$$\dot{y} = \frac{45}{2}e^{-4t} - \frac{5}{2}$$

$$\therefore \mathbf{v}(t) = \begin{pmatrix} 20\sqrt{3}\,e^{-4t} \\ \frac{45}{2}e^{-4t} - \frac{5}{2} \end{pmatrix}$$

c) Show that $\mathbf{r}(t) = \begin{pmatrix} 5\sqrt{3}\,(1-e^{-4t}) \\ \frac{45}{8}(1-e^{-4t}) - \frac{5}{2}t \end{pmatrix}$

$$\frac{dx}{dt} = 20\sqrt{3}\,e^{-4t}$$

$$\int_0^x dx = 20\sqrt{3}\int_0^t e^{-4t}\,dt$$

$$x = -5\sqrt{3}\left[e^{-4t}\right]_0^t$$

$$= -5\sqrt{3}\,(e^{-4t}-1)$$

$$= 5\sqrt{3}\,(1-e^{-4t})$$

$$\frac{dy}{dt} = \frac{45}{2}e^{-4t} - \frac{5}{2}$$

$$\int_0^y dy = \int_0^t \left(\frac{45}{2}e^{-4t} - \frac{5}{2}\right)dt$$

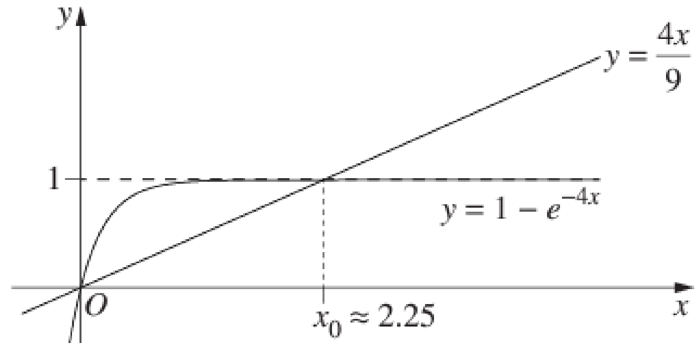
$$y = \left[-\frac{45}{8}e^{-4t} - \frac{5}{2}t\right]_0^t$$

$$= -\frac{45}{8}e^{-4t} - \frac{5}{2}t + \frac{45}{8}$$

$$= \frac{45}{8}(1-e^{-4t}) - \frac{5}{2}t$$

$$\therefore \mathbf{r}(t) = \begin{pmatrix} 5\sqrt{3}\,(1-e^{-4t}) \\ \frac{45}{8}(1-e^{-4t}) - \frac{5}{2}t \end{pmatrix}$$

d) The graphs $y = 1 - e^{-4x}$ and $y = \frac{4x}{9}$ are given in the diagram below



Using the diagram, find the horizontal range of the particle, giving your answer rounded to one decimal place.

Exercise 6F;
2, 4, 5, 8, 10, 11,
12, 14, 15, 16, 17, 19, 20

Exercise 6G;
2, 3, 5, 6, 7, 9, 12, 13, 16

particle hits the ground when $\mathbf{r}(t) = \begin{pmatrix} x \\ 0 \end{pmatrix}$

i.e. $\frac{45}{8}(1 - e^{-4t}) - \frac{5}{2}t = 0$

$$\frac{45}{8}(1 - e^{-4t}) = \frac{5}{2}t$$

$$1 - e^{-4t} = \frac{4t}{9}$$

from the diagram; $t \approx 2.25$

when $t = 2.25$, $x = 5\sqrt{3}(1 - e^{-4(2.25)})$

$$= 8.659185278\ldots$$

$\therefore$ the range of the particle is 8.7 metres (to 1 dp)