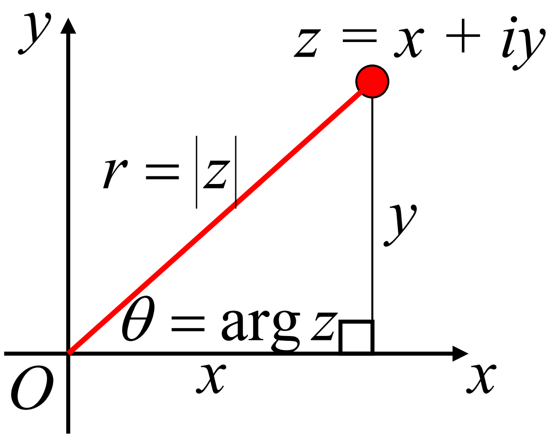
Mod-Arg Form



NOTE:
Arg z is the **principal argument**
arg z is the **arbitrary argument**

Modulus

The modulus of a complex number is the length of the vector OZ

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

$$|z| = \sqrt{x^2 + y^2}$$

Argument

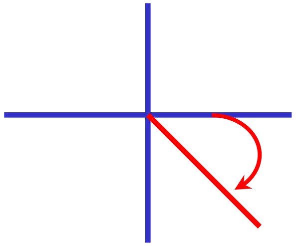
The argument of a complex number is the angle the vector OZ makes with the positive real (x) axis

$$\arg z = \tan^{-1}\left(\frac{y}{x}\right) \qquad -\pi < \text{Arg } z \leq \pi$$

$e.g.$ Find the modulus and argument of $4-4i$

$$|4-4i| = \sqrt{4^2+(-4)^2}$$
$$= \sqrt{32}$$
$$= 4\sqrt{2}$$

$$\arg(4-4i) = \tan^{-1}\left(\frac{-4}{4}\right)$$
$$= \tan^{-1}(-1)$$
$$= -\frac{\pi}{4}$$



Every complex number can be written in terms of its modulus and argument

$$z = x + iy$$
$$= r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

The **mod-arg** form of z is;

$$z = r(\cos\theta + i\sin\theta)$$
$$z = rcis\theta$$

where; $r = |z|$

$$\theta = \arg z$$

$$e.g.\ (i)\ 4-4i \ = 4\sqrt{2}cis\left(-\frac{\pi}{4}\right)$$

$$(ii)\ \sqrt{3}+i$$

$$|z| = \sqrt{\left(\sqrt{3}\right)^2+1^2}$$
$$= \sqrt{4}$$
$$= 2$$

$$\arg z = \tan^{-1}\frac{1}{\sqrt{3}}$$
$$= \frac{\pi}{6}$$

$$\therefore \sqrt{3}+i = 2cis\frac{\pi}{6}$$

(ii) Convert $6\text{cis}\frac{\pi}{6}$ to Cartsian form

$$6\text{cis}\frac{\pi}{6} = 6(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$$
$$= 6(\frac{\sqrt{3}}{2} + \frac{1}{2}i)$$
$$= 3\sqrt{3} + 3i$$

Mod-Arg Relations

$$(1)\ |z_1 z_2| = |z_1||z_2|$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

NOTE: Multiplication rotates z_1 by $\arg z_2$

Proof: let $z_1 = r_1 cis\theta_1$ and $z_2 = r_2 cis\theta_2$

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\times r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 - \sin\theta_1\sin\theta_2)$$

$$= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\}$$

$$= r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$$

$$\therefore |z_1 z_2| = r_1 r_2 \qquad \arg(z_1 z_2) = \theta_1 + \theta_2$$

$$= |z_1||z_2| \qquad = \arg z_1 + \arg z_2$$

NOTE: it follows that;

$$|z_1 z_2 z_3 \ldots z_n| = |z_1||z_2||z_3|\ldots|z_n|$$

$$\arg(z_1 z_2 z_3 \ldots z_n) = \arg z_1 + \arg z_2 + \arg z_3 + \ldots + \arg z_n$$

$$(2)\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

NOTE: it follows that;

$$\left|\frac{z_1 z_2}{z_3 z_4}\right| = \frac{|z_1||z_2|}{|z_3||z_4|}$$

$$\arg\left(\frac{z_1 z_2}{z_3 z_4}\right) = \arg z_1 + \arg z_2 - \arg z_3 - \arg z_4$$

$$(3)\left|z^n\right| = |z|^n$$

$$\arg\left(z^n\right) = n \arg z$$

e.g. Find the modulus and argument of $z = \dfrac{(5+i)(-2-i)}{3+2i}$

$$|z| = \frac{\sqrt{5^2+1^2}\sqrt{(-2)^2+(-1)^2}}{\sqrt{3^2+2^2}}$$
$$= \frac{\sqrt{26}\sqrt{5}}{\sqrt{13}}$$
$$= \underline{\sqrt{10}}$$

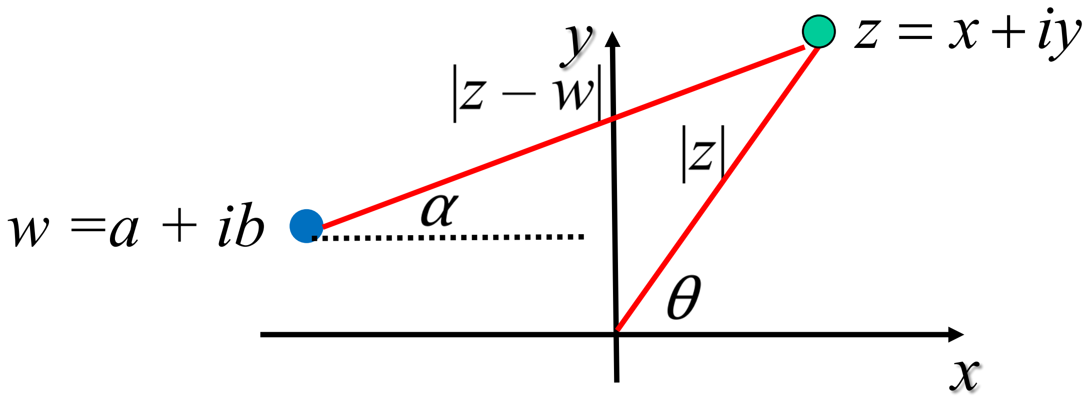
$$\arg z = \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{-1}{-2}\right) - \tan^{-1}\left(\frac{2}{3}\right)$$
$$= 11^\circ 19' + \left(-153^\circ 26'\right) - 33^\circ 41'$$
$$= \underline{-175^\circ 48'}$$

Shifting the Reference Point

In the Cartesian plane, the reference point is sometimes moved from the origin to another point

e.g. $(x-1)^2 + (y-2)^2 = 1$, the centre of the circle has moved from the origin to (1,2)

A similar process is used in the Argand Diagram



$|z|$ = distance from z to the origin

$\arg z$ = direction of the position vector $\overrightarrow{OZ}$ (θ)

$|z-w|$ = distance from z to w

$\arg(z-w)$ = direction of the displacement vector $\overrightarrow{WZ}$ (α)

Exercise 1D; 1ace, 2acf, 3bdf, 4ac, 5bcf, 6, 7adf, 8ace, 9ace, 13, 15, 16, 17, 18, 20, 21, 22, 24