Integration By Parts

When an integral is a product of two functions and neither is the derivative of the other, we integrate by parts.

$$\int u\,dv = uv - \int v\,du$$

Proof:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$uv = \int u\,dv + \int v\,du$$

$$\int u\,dv = uv - \int v\,du$$

u should be chosen so that differentiation makes it a simpler function.

dv should be chosen so that it can be integrated e.g. (*i*) $\int x\sin x dx$

$u = x$ $\quad v = -\cos x$

$du = dx$ $\quad dv = \sin x dx$

$$= -x\cos x + \int \cos x dx$$

$$= \sin x - x\cos x + c$$

(*ii*) $\int \log x dx$

$u = \log x$ $\quad v = x$

$du = \dfrac{dx}{x}$ $\quad dv = dx$

$$= x\log x - \int dx$$

$$= x\log x - x + c$$

$$(iii)\int_0^1 xe^{-7x}dx \qquad u = x \qquad v = -\frac{1}{7}e^{-7x}$$

$$du = dx \qquad dv = e^{-7x}dx$$

$$= \left[-\frac{1}{7}xe^{-7x}\right]_0^1 + \frac{1}{7}\int_0^1 e^{-7x}dx$$

$$= \left[-\frac{1}{7}xe^{-7x} - \frac{1}{49}e^{-7x}\right]_0^1$$

$$= \left\{-\frac{1}{7}e^{-7} - \frac{1}{49}e^{-7}\right\} - \left\{0 - \frac{1}{49}\right\}$$

$$= -\frac{8}{49}e^{-7} + \frac{1}{49}$$

$$(iv)\int e^x \cos x\,dx$$

$$u = e^x \qquad v = \sin x$$

$$du = e^x dx \qquad dv = \cos x\,dx$$

$$= e^x \sin x - \int e^x \sin x\,dx$$

$$u = e^x \qquad v = -\cos x$$

$$du = e^x dx \qquad dv = \sin x\,dx$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$$

$$\therefore 2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$

$$\underline{\int e^x \cos x\,dx = \frac{1}{2}e^x \sin x + \frac{1}{2}e^x \cos x + c}$$

(v) 2024 Extension 2 HSC Question 14 d)

The following argument attempts to prove that $0 = 1$

We evaluate $\int \frac{1}{x}\,dx$ using the method of integration by parts.

$$\int \frac{1}{x}\,dx = \int \frac{1}{x} \times 1\,dx$$

$$= \frac{1}{x} \times x - \int -\frac{1}{x^2} x dx$$

$$= 1 + \int \frac{1}{x}\,dx$$

So we have

$$\int \frac{1}{x}\,dx = 1 + \int \frac{1}{x}\,dx$$

We may now subtract $\int \frac{1}{x}\,dx$ from both sides to show that $0 = 1$

Explain what is wrong with this argument.

proof should conclude

$$\int \frac{1}{x}\, dx = 1 + \int \frac{1}{x}\, dx$$

$$\int 0\, dx = 1$$

$$c = 1$$

i.e. when 0 is integrated you get a constant of integration which may or may not be 0.
In this case the constant of integration is 1.

Exercise 4F; 1abdf, 2cef, 3c, 4c, 5bc, 6bc, 7ac, 8b, 9a, 10ac, 11bc, 12, 13acd, 14ac, 15a, 16b, 17