Trig Substitutions

$\sqrt{a^2 + x^2}$ use $x = a\tan\theta$

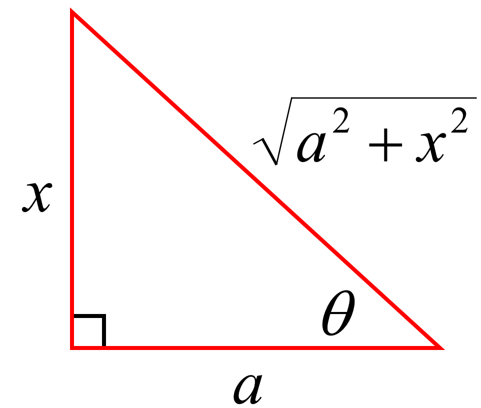
$\sqrt{a^2 - x^2}$ use $x = a\sin\theta$ or $x = a\cos\theta$

$\sqrt{x^2 - a^2}$ use $x = a\sec\theta$

$$\text{e.g. }(i)\int \frac{1}{\sqrt{a^2+x^2}}\,dx$$

$$x = a\tan\theta$$

$$dx = a\sec^2\theta\, d\theta$$



$$= \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2+a^2\tan^2\theta}}$$

$$= \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2\sec^2\theta}}$$

$$= \int \sec\theta\, d\theta$$

$$= \log|\sec\theta + \tan\theta| + c$$

$$= \log\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + c$$

$$= \log\left|\sqrt{a^2+x^2} + x\right| - \log|a| + c$$

$$= \log\left|\sqrt{a^2+x^2} + x\right| + c$$

$$(ii)\int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$x = \sin\theta$$

$$dx = \cos\theta\, d\theta$$

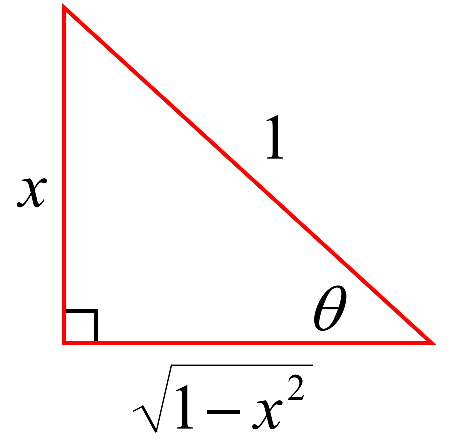
$$= \int \frac{\sin\theta \cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \int \frac{\sin\theta \cos\theta\, d\theta}{\sqrt{\cos^2\theta}}$$

$$= \int \sin\theta\, d\theta$$

$$= -\cos\theta + c$$

$$= -\sqrt{1-x^2} + c$$



$$(iii)\int\sqrt{x^2+3}\,dx$$

$$x=\sqrt{3}\tan\theta$$

$$dx=\sqrt{3}\sec^2\theta\,d\theta$$

$$=\int\left(\sqrt{3}\sec\theta\right)\left(\sqrt{3}\sec^2\theta\right)d\theta$$

$$=3\int\sec^3\theta\,d\theta$$

$$u=\sec\theta \qquad v=\tan\theta$$

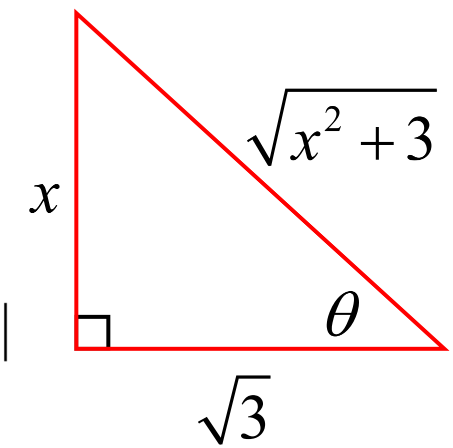
$$du=\sec\theta\tan\theta\,d\theta \qquad dv=\sec^2\theta\,d\theta$$

$$=3\int\sec\theta\sec^2\theta\,d\theta$$

$$=3\sec\theta\tan\theta-3\int\sec\theta\tan^2\theta\,d\theta$$

$$=3\sec\theta\tan\theta-3\int\sec^3\theta\,d\theta+3\int\sec\theta$$

$$=3\sec\theta\tan\theta-3\int\sec^3\theta\,d\theta+3\log|\sec\theta+\tan\theta|$$



$$=3\frac{\sqrt{x^2+3}}{\sqrt{3}}\times\frac{x}{\sqrt{3}}-\int\sqrt{x^2+3}\,dx+3\log\left|\frac{\sqrt{x^2+3}}{\sqrt{3}}+\frac{x}{\sqrt{3}}\right|$$

$$\therefore 2\int \sqrt{x^2 + 3}\,dx = x\sqrt{x^2 + 3} + 3\log\left| \frac{\sqrt{x^2 + 3}}{\sqrt{3}} + \frac{x}{\sqrt{3}} \right| + c$$

$$\int \sqrt{x^2 + 3}\,dx = \frac{x\sqrt{x^2 + 3}}{2} + \frac{3}{2}\log\left| \frac{\sqrt{x^2 + 3}}{\sqrt{3}} + \frac{x}{\sqrt{3}} \right| + c$$

$$\int \sqrt{x^2 + 3}\,dx = \frac{x\sqrt{x^2 + 3}}{2} + \frac{3}{2}\log\left| \sqrt{x^2 + 3} + x \right| + c$$

(iv) 2024 Extension 2 HSC Question 15 d)

Using a suitable substitution, find $\displaystyle\int \frac{2x^2}{\sqrt{2x-x^2}}dx$

$$x-1=\sin\theta$$
$$dx=\cos\theta\, d\theta$$

$$\int \frac{2x^2}{\sqrt{2x-x^2}}dx=\int \frac{2x^2}{\sqrt{1-(x-1)^2}dx}$$

$$=2\int \frac{(1+\sin\theta)^2}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta$$

$$=2\int (1+2\sin\theta+\sin^2\theta)d\theta$$

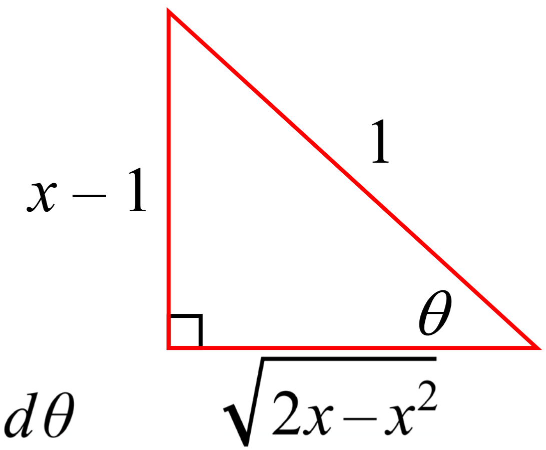
$$=2\int \left(1+2\sin\theta+\frac{1}{2}(1-\cos 2\theta)\right)d\theta$$

$$=\int (3+4\sin\theta-\cos 2\theta)d\theta$$

$$=3\theta-4\cos\theta-\frac{1}{2}\sin 2\theta+c$$

$$=3\theta-4\cos\theta-\sin\theta\cos\theta+c$$

$$=3\sin^{-1}(x-1)-4\sqrt{2x-x^2}-(x-1)\sqrt{2x-x^2}+c$$



Patel
Exercise 2E;
1, 2, 3, 5,
6, 7, 9, 13,
17, 19, 20