Geometric Series

An geometric series is a sequence of numbers in which each term after the first is found by **multiplying** a constant amount to the previous term.

The constant amount is called the **common ratio**, symbolised, ***r***.

$$r = \frac{T_2}{a} = \frac{T_3}{T_2}$$

$$r = \frac{T_n}{T_{n-1}}$$

$$T_n = rT_{n-1}$$

(recursive formula)

$$T_1 = a$$
$$T_2 = ar$$
$$T_3 = ar^2$$
$$T_n = ar^{n-1}$$

When plotted on a number plane, the graph of a geometric sequence is an exponential function

$e.g.(i)$ Find r and the general term of $2, 8, 32, \ldots$

$$a = 2, r = 4$$

$$\begin{aligned} T_n &= ar^{n-1} \\ &= 2(4)^{n-1} \\ &= 2(2^2)^{n-1} \\ &= 2(2)^{2n-2} \end{aligned}$$

$$\therefore T_n = 2^{2n-1}$$

(*ii*) If $T_2 = 7$ and $T_4 = 49$, find the general term

$$ar = 7$$
$$ar^3 = 49$$

$$r^2 = 7$$
$$r = \pm\sqrt{7} \quad \therefore a = \pm\sqrt{7}$$
$$T_n = (\sqrt{7})(\sqrt{7})^{n-1}$$
$$= (\sqrt{7})^n$$

OR

$$T_n = (-\sqrt{7})(-\sqrt{7})^{n-1}$$
$$= (-\sqrt{7})^n$$
$$= (-1)^n(\sqrt{7})^n$$

(*iii*) find the first term of 1, 4, 16, … to be greater than 500.

$$a = 1, r = 4 \qquad T_n = 1(4)^{n-1}$$
$$T_n > 500$$
$$4^{n-1} > 500$$
$$\log 4^{n-1} > \log 500$$
$$(n-1)\log 4 > \log 500$$
$$n - 1 > 4.48$$
$$n > 5.48$$
$$T_6 = 1024, \text{is the first term} > 500$$

Arithmetic & Geometric Means

Arithmetic Mean(average)

$$AM = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$$

Geometric Mean

$$GM = \sqrt[n]{a_1 a_2 a_3 \ldots a_n}$$

Note: all the values must be positive to find a geometric mean

*e.g.*Find the *AM* and *GM* of 4 and 25

$$AM = \frac{25+4}{2} = \frac{29}{2}$$

$$GM = \sqrt{25 \times 4} = \sqrt{100} = 10$$

(ii) Find x and y if 2, x, y, 128 form a GP

$$x = \sqrt{2y} \qquad\qquad y = \sqrt{128x}$$

$$y^2 = 128x$$

$$= 128\sqrt{2y}$$

$$y^4 = 32768y$$

$$y(y^3 - 32768) = 0$$

$$y = 0 \quad \text{or} \quad y = 32$$

$$\therefore y = 32 \quad \text{(0 cannot be a term in a GP)}$$

$$\underline{x = 8\ ,\ y = 32}$$

Some Other Means

Harmonic Mean

$$HM = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_n}}$$

The arithmetic mean, geometric mean and the harmonic mean are the three **Pythagorean means**

Quadratic Mean

$$QM = \sqrt{\frac{a_1^{\,2} + a_2^{\,2} + a_3^{\,2} + \dots + a_n^{\,2}}{n}}$$

$$QM \geq AM \geq GM \geq HM$$

Note: equality occurs when all of the numbers are equal

eg (*i*) Prove $a + b \geq 2\sqrt{ab}$

start with a known fact

$$(a-b)^2 \geq 0$$

$$a^2 - 2ab + b^2 \geq 0$$

$$a^2 + b^2 \geq 2ab$$

substitute expressions into a known inequality

let $a = \sqrt{a}$ and $b = \sqrt{b}$

$$\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 \geq 2\left(\sqrt{a}\right)\left(\sqrt{b}\right)$$

$$a + b \geq 2\sqrt{ab}$$

this proves that AM $\geq$ GM for two numbers

(ii) Prove $\sqrt{ab} \geq \dfrac{2}{\dfrac{1}{a}+\dfrac{1}{b}}$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

AM ≥ GM

ask yourself

how can I turn what I've got into what I want?

$$\frac{a+b}{2} \times \frac{2\sqrt{ab}}{a+b} \geq \sqrt{ab} \times \frac{2\sqrt{ab}}{a+b}$$

whilst you **cannot** solve like an inequation

you **can** use algebraic manipulation of a known inequality to get the desired result

$$\sqrt{ab} \geq \frac{2ab}{a+b}$$

$$= \frac{2}{\dfrac{a+b}{ab}}$$

once one side is what I want, it **must** be possible to manipulate the other side

$$= \frac{2}{\dfrac{1}{b}+\dfrac{1}{a}}$$

$$\therefore \sqrt{ab} \geq \frac{2}{\dfrac{1}{a}+\dfrac{1}{b}}$$

this proves that GM ≥ HM for two numbers

(iii) Prove $\sqrt{\dfrac{a^2+b^2}{2}} \geq \dfrac{a+b}{2}$

$$a^2+b^2 \geq 2ab$$

AM ≥ GM

$$2a^2+2b^2 \geq 2ab+a^2+b^2$$

$$\frac{a^2+b^2}{2} \geq \frac{(a+b)^2}{4}$$

$$= \left(\frac{a+b}{2}\right)^2$$

always explain why you can make "non-obvious" claims

$$\therefore \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$$

$y=\sqrt{x}$ is a continually increasing function

this proves that QM ≥ AM for two numbers

Exercise 1C; 4be, 6, 8cf, 9ad, 10f, 13, 14, 16c, 19b

Exercise 1D; 1ae, 2af, 3ace etc, 4 (*use AM & GM*), 5b, 6b, 9, 10a, 11, 12, 13bd, 14, 16, 18ab, 19, 20