Reduction Formula

Reduction (or recurrence) formulae can be used when the integrand is raised to a power.

Integration by parts often used to find the formula.

e.g. (i) Find $\displaystyle\int_1^e x(\log x)^3\, dx$

$$I_n = \int_1^e x(\log x)^n\, dx$$

$$u = \left(\log x\right)^n \qquad\qquad v = \frac{1}{2}x^2$$

$$du = \frac{n\left(\log x\right)^{n-1}}{x}\, dx \qquad dv = x\, dx$$

$$= \frac{1}{2}\left[\, x^2\left(\log x\right)^n\,\right]_1^e - \frac{n}{2}\int_1^e x(\log x)^{n-1}\, dx$$

$$= \frac{e^2}{2} - \frac{n}{2}\int_1^e x(\log x)^{n-1}\, dx$$

$$= \frac{e^2}{2} - \frac{n}{2}I_{n-1}$$

$$\therefore \int_1^e x(\log x)^3\, dx = I_3 = \frac{e^2}{2} - \frac{3}{2}I_2 \qquad \textbf{\textit{OR}} \qquad I_0 = \int_1^e x\, dx = \frac{1}{2}\left[x^2\right]_1^e$$

$$= \frac{e^2}{2} - \frac{3}{2}\left(\frac{e^2}{2} - I_1\right) \qquad\qquad = \frac{1}{2}\left(e^2 - 1\right)$$

$$= -\frac{e^2}{4} + \frac{3}{2}I_1 \qquad\qquad I_1 = \frac{e^2}{2} - \frac{1}{2}I_0 = \frac{e^2}{2} - \frac{1}{2}\times\frac{1}{2}\left(e^2 - 1\right)$$

$$= -\frac{e^2}{4} + \frac{3}{2}\left(\frac{e^2}{2} - \frac{1}{2}I_0\right) \qquad\qquad = \frac{e^2}{4} + \frac{1}{4}$$

$$= \frac{e^2}{2} - \frac{3}{4}I_0 \qquad\qquad I_2 = \frac{e^2}{2} - I_1 = \frac{e^2}{2} - \frac{1}{4}\left(e^2 + 1\right)$$

$$= \frac{e^2}{2} - \frac{3}{4}\int_1^e x\, dx \qquad\qquad = \frac{e^2}{4} - \frac{1}{4}$$

$$= \frac{e^2}{2} - \frac{3}{8}\left[x^2\right]_1^e \qquad\qquad I_3 = \frac{e^2}{2} - \frac{3}{2}I_2 = \frac{e^2}{2} - \frac{3}{2}\times\frac{1}{4}\left(e^2 - 1\right)$$

$$= \frac{e^2}{2} - \frac{3e^2}{8} + \frac{3}{8} \quad = \frac{e^2}{8} + \frac{3}{8} \qquad\qquad = \frac{e^2}{8} + \frac{3}{8}$$

Integration by parts is the commonest way of getting reduction formulae, but it is not the only method.

Some just involve the use of a **trig identity**.

(ii) Given that $I_n = \int \cot^n x \, dx$, find I_6

$$I_n = \int \cot^n x \, dx$$

$$= \int \cot^{n-2} x \cot^2 x \, dx$$

$$= \int \cot^{n-2} x \left(\mathrm{cosec}^2 x - 1 \right) dx$$

$$= \int \cot^{n-2} x \, \mathrm{cosec}^2 x \, dx - \int \cot^{n-2} x \, dx$$

$$= -\int u^{n-2} du - I_{n-2}$$

$$= -\frac{1}{n-1} u^{n-1} - I_{n-2}$$

$$= -\frac{1}{n-1} \cot^{n-1} x - I_{n-2}$$

$\int \cot^n x \, dx$

Uses the same technique, for all powers.

So just use that technique

$$u = \cot x$$

$$du = -\mathrm{cosec}^2 x \, dx$$

$$\int \cot^6 x \, dx = I_6$$

$$= -\frac{1}{5}\cot^5 x - I_4$$

$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x + I_2$$

$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - I_0$$

$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - \int dx$$

$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - x + c$$

If you see a reduction formula in the form
$$I_n + bI_{n-k} + c = 0$$
and b is independent of n,
then a simple addition approach is likely to work

this idea can be useful in show that questions

2025 Extension 2 HSC Question 14 a) (i)

Let $I_n = \displaystyle\int_{\pi}^{\frac{\pi}{2}} \cot^{2n}\theta \, d\theta$ for integers $n \geq 0$

Show that $I_n = \dfrac{1}{2n-1} - I_{n-1}$ for $n > 0$, given that $\dfrac{d}{d\theta}\cot\theta = -\operatorname{cosec}^2\theta$

$$I_n + I_{n-1} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot^{2n}\theta + \cot^{2n-2}\theta)\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{2n-2}\theta(\ \cot^2\theta + 1)d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{2n-2}\theta \operatorname{cosec}^2\theta d\theta$$

$$= -\frac{1}{2n-1}\left[\cot^{2n-1}\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= -\frac{1}{2n-1}(0-1)$$

$$= \frac{1}{2n-1}$$

$$\therefore I_n = \frac{1}{2n-1} - I_{n-1}$$

Sometimes integration by parts is not enough by itself.

Some involve the use of **parts** along with **algebraic manipulation** or use of a **trig identity**.

(iii) Given that $I_n = \int \left(1+x^2\right)^n dx,$ find a reduction formula

  connecting I_n with I_{n-1}

$$I_n = \int \left(1+x^2\right)^n dx$$

$$u = \left(1+x^2\right)^n \qquad v = x$$

$$= x\left(1+x^2\right)^n - 2n\int x^2 \left(1+x^2\right)^{n-1} dx \qquad du = 2nx\left(1+x^2\right)^{n-1} dx \quad dv = dx$$

$$= x\left(1+x^2\right)^n - 2n\int \left(1+x^2-1\right)\left(1+x^2\right)^{n-1} dx$$

$$= x\left(1+x^2\right)^n - 2n\int \left(1+x^2\right)\left(1+x^2\right)^{n-1} dx + 2n\int \left(1+x^2\right)^{n-1} dx$$

$$= x\left(1+x^2\right)^n - 2n\int \left(1+x^2\right)^n dx + 2n\int \left(1+x^2\right)^{n-1} dx$$

$$= x\left(1+x^2\right)^n - 2nI_n + 2nI_{n-1}$$

$$\left(2n+1\right)I_n = x\left(1+x^2\right)^n + 2nI_{n-1} \qquad \therefore I_n = \frac{x\left(1+x^2\right)^n}{\left(2n+1\right)} + \frac{2n}{\left(2n+1\right)}I_{n-1}$$

e.g. (*iv*) (1987)

Given that $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$, prove that $I_n = \left(\dfrac{n-1}{n}\right) I_{n-2}$

where n is an integer and $n \geq 2$, hence evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x \, dx$

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x \, dx$$

$u = \cos^{n-1} x$ $v = \sin x$

$du = -(n-1)\cos^{n-2} x \sin x \, dx$ $dv = \cos x \, dx$

$$= \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, dx$$

$$= \left\{\cos^{n-1}\frac{\pi}{2}\sin\frac{\pi}{2} - \cos^{n-1} 0 \sin 0\right\} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x\left(1 - \cos^2 x\right) dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx - (n-1)\int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

$$= (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore nI_n = (n-1)I_{n-2}$$

$$I_n = \left(\frac{n-1}{n}\right)I_{n-2}$$

$$\int_0^{\frac{\pi}{2}} \cos^5 x \, dx = I_5$$

$$= \frac{4}{5} I_3$$

$$= \frac{4}{5} \times \frac{2}{3} I_1$$

$$= \frac{8}{15} \int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$= \frac{8}{15} \left[\sin x \right]_0^{\frac{\pi}{2}}$$

$$= \frac{8}{15} \left(\sin \frac{\pi}{2} - \sin 0 \right)$$

$$= \frac{8}{15}$$

(*v*) **2022 Extension 2 HSC Question 14 b)**

Let $J_n = \displaystyle\int_0^1 x^n e^{-x}\,dx$, where n is a non-negative integer

(*i*) Show that $J_0 = 1 - \dfrac{1}{e}$

$$J_0 = \int_0^1 e^{-x}\,dx$$

$$= -\left[e^{-x}\right]_0^1$$

$$= -e^{-1} + e^0$$

$$= 1 - \dfrac{1}{e}$$

(ii) Show that $J_n \leq \dfrac{1}{n+1}$

$$0 \leq x \leq 1$$

$$\frac{1}{e} \leq e^{-x} \leq 1 \qquad\qquad (e^{-x} \text{ is continually decreasing})$$

$$J_n = \int_0^1 x^n e^{-x}\,dx \;\leq\; \int_0^1 x^n\,dx$$

$$= \frac{1}{n+1}\left[x^{n+1}\right]_0^1$$

$$= \frac{1}{n+1}(1-0)$$

$$= \frac{1}{n+1}$$

$$\therefore J_n \leq \frac{1}{n+1}$$

(iii) Show that $J_n = nJ_{n-1} - \dfrac{1}{e}$, for $n \geq 1$

$$J_n = \int_0^1 x^n e^{-x} dx \qquad\qquad u = x^n \qquad\qquad v = -e^{-x}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad du = nx^{n-1} dx \qquad dv = e^{-x} dx$$

$$= \left[-x^n e^{-x} \right]_0^1 + n \int_0^1 x^{n-1} e^{-x} dx$$

$$= -e^{-1} + 0 + nJ_{n-1}$$

$$= nJ_{n-1} - \frac{1}{e}$$

(iv) Using parts (i) and (iii), show by mathematical induction, or otherwise that for all $n \geq 0$

$$J_n = n! - \frac{n!}{e} \sum_{r=0}^{n} \frac{1}{r!}$$

Show $J_n = n! - \dfrac{n!}{e} \displaystyle\sum_{r=0}^{n} \dfrac{1}{r!}$

Prove the result is true for $n = 0$

$$J_0 = 0! - \dfrac{0!}{e} \sum_{r=0}^{0} \dfrac{1}{r!}$$

$$= 1 - \dfrac{1}{e}$$

Hence the result is true for $n = 0$

Assume the result is true for $n = k$

$$\text{i.e } J_k = k! - \dfrac{k!}{e} \sum_{r=0}^{k} \dfrac{1}{r!}$$

Prove the result is true for $n = k + 1$

$$\text{i.e Prove } J_{k+1} = (k+1)! - \dfrac{(k+1)!}{e} \sum_{r=0}^{k+1} \dfrac{1}{r!}$$

Proof:

$$J_{k+1} = (k+1)J_k - \frac{1}{e}$$

$$= (k+1)\left[k! - \frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}\right] - \frac{1}{e} \qquad \text{(by assumption)}$$

$$= (k+1)! - \frac{k+1!}{e}\sum_{r=0}^{k}\frac{1}{r!} - \frac{1}{e}$$

$$= (k+1)! - \frac{k+1!}{e}\sum_{r=0}^{k}\frac{1}{r!} - \frac{(k+1)!}{e} \times \frac{1}{(k+1)!}$$

$$= (k+1)! - \frac{(k+1)!}{e}\sum_{r=0}^{k+1}\frac{1}{r!}$$

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 0$, then it is true $\forall n \in \mathbb{Z} : n \geq 0$ by induction

(*v*) Using parts (ii) and (iv) prove that $e = \lim\limits_{n \to \infty} \sum\limits_{r=0}^{n} \dfrac{1}{r!}$

$$0 \le J_n \le \frac{1}{n+1}$$

$$0 \le n! - \frac{n!}{e} \sum_{r=0}^{n} \frac{1}{r!} \le \frac{1}{n+1}$$

$$0 \le e - \sum_{r=0}^{n} \frac{1}{r!} \le \frac{e}{(n+1)!}$$

$$0 \le \lim_{n \to \infty} \left(e - \sum_{r=0}^{n} \frac{1}{r!} \right) \le \lim_{n \to \infty} \frac{e}{(n+1)!}$$

$$0 \le e - \lim_{n \to \infty} \sum_{r=0}^{n} \frac{1}{r!} \le 0$$

$$\therefore e - \lim_{n \to \infty} \sum_{r=0}^{n} \frac{1}{r!} = 0$$

$$\lim_{n \to \infty} \sum_{r=0}^{n} \frac{1}{r!} = e$$

Exercise 4H;
2, 5, 6, 7, 8, 9, 10, 12, 14, 17