

Using Symmetry

(1) Even

$$f(-x) = f(x)$$

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

NOTE: horizontal shift

$$\int_{c-a}^{c+a} f(x-c) dx = 2 \int_c^{c+a} f(x-c) dx$$

(2) Odd

$$f(-x) = -f(x)$$

$$\int_{-a}^a f(x) dx = 0$$

NOTE: horizontal shift

$$\int_{c-a}^{c+a} f(x-c) dx = 0$$

(3)

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

Proof:

$$\begin{aligned} & \int_0^a f(a-x)dx \\ &= -\int_a^0 f(u)du \\ &= \int_0^a f(u)du \\ &= \int_0^a f(x)dx \end{aligned}$$

$$u = a - x$$

$$du = -dx$$

$$x = 0, u = a$$

$$x = a, u = 0$$

NOTE:

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

odd \times odd = even

odd \times even = odd

even \times even = even

$$\text{e.g. (i) } \int_{-1}^1 \sin^3 x dx = \underline{0} \quad (\text{odd function})^3 = \text{odd function}$$

$$\begin{aligned} \text{(ii) } \int_0^1 x^2 \sqrt{1-x} dx &= \int_0^1 (1-x)^2 \sqrt{x} dx \\ &= \int_0^1 \left(x^{\frac{1}{2}} - 2x^{\frac{3}{2}} + x^{\frac{5}{2}} \right) dx \\ &= \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{4}{5} x^{\frac{5}{2}} + \frac{2}{7} x^{\frac{7}{2}} \right]_0^1 \\ &= \frac{2}{3} - \frac{4}{5} + \frac{2}{7} - 0 \\ &= \underline{\underline{\frac{16}{105}}} \end{aligned}$$

Improper Integrals

An **improper** integral is a definite integral where the integrand is undefined at some point in the interval or unbounded.

We must use limits to solve, if a solution exists.

$$\begin{aligned} \text{e.g. (i)} \quad \int_0^1 \frac{dx}{x} &= \lim_{N \rightarrow 0} \int_N^1 \frac{dx}{x} \\ &= \lim_{N \rightarrow 0} [\log x]_N^1 \\ &= \lim_{N \rightarrow 0} (-\log N) \end{aligned}$$

Undefined

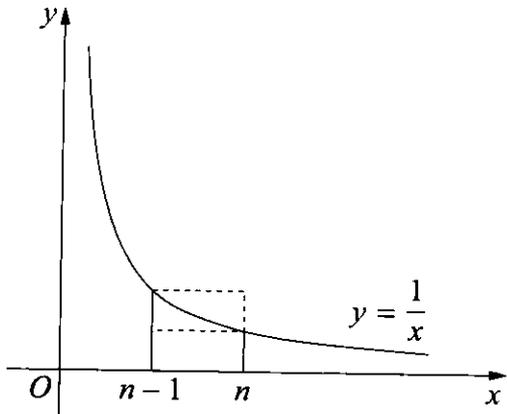
\therefore integral is undefined

$$\begin{aligned} \text{(ii)} \quad \int_1^{\infty} x e^{-x^2} dx &= \lim_{N \rightarrow \infty} \int_1^N x e^{-x^2} dx \\ &= \lim_{N \rightarrow \infty} -\frac{1}{2} [e^{-x^2}]_1^N \\ &= \lim_{N \rightarrow \infty} -\frac{1}{2} (e^{-N^2} - e^{-1}) \\ &= \frac{1}{2e} \end{aligned}$$

Area and Inequalities

$$\text{If } g(x) \leq f(x) \leq h(x) \text{ for } a \leq x \leq b$$
$$\text{then } \int_a^b g(x)dx \leq \int_a^b f(x)dx \leq \int_a^b h(x)dx$$

e.g. (2009 Question 8 b)



Let n be a positive integer greater than 1.

The area of the region under the curve $y = \frac{1}{x}$ from $x = n - 1$ to $x = n$ is between the area of two rectangles, as shown in the diagram.

$$\text{Show that } e^{-\frac{n}{n-1}} < \left(1 - \frac{1}{n}\right)^n < e^{-1}$$

Area of inner rectangle < Area under the curve < Area of outer rectangle

$$(1) \left(\frac{1}{n} \right) < \int_{n-1}^n \frac{dx}{x} < (1) \left(\frac{1}{n-1} \right)$$

$$\frac{1}{n} < [\log x]_{n-1}^n < \frac{1}{n-1}$$

$$\frac{1}{n} < \log \left(\frac{n}{n-1} \right) < \frac{1}{n-1}$$

$$1 < n \log \left(\frac{n}{n-1} \right) < \frac{n}{n-1}$$

$$1 < \log \left(\frac{n}{n-1} \right)^n < \frac{n}{n-1}$$

as $y = e^x$ is continually increasing

$$e < \left(\frac{n}{n-1} \right)^n < e^{\frac{n}{n-1}}$$

$$\frac{1}{e} > \left(\frac{n-1}{n} \right)^n > \frac{1}{e^{\frac{n}{n-1}}}$$

$$\text{i.e. } e^{-\frac{n}{n-1}} < \left(1 - \frac{1}{n} \right)^n < e^{-1}$$

***Old Cambridge Exercise 2I; 1, 2ac, 3, 6, 8, 9c, 10,
11ad, 12c, 13, 18***

Note: $(2a - x)$ instead of $(a - x)$

Exercise 4I;1, 2defgh, 3a to h, 4, 6, 7b, 8, 12, 15

The 100 (*not 78*)